Robust Mean-Conditional Value at Risk Portfolio Optimization

Farzaneh Piri, Maziar Salahi, Farshid Mehrdoust

ABSTRACT

In the portfolio optimization, the goal is to distribute the fixed capital on a set of investment opportunities to maximize return while managing risk. Risk and return are quantities that are used as input parameters for the optimal allocation of the capital in the suggested models. But these quantities are not known at the time of the formulation and solving the problem. Thus they should be estimated to solve the problem which might lead to large error. One of the widely used approaches to deal with such a situation, is robust optimization. In this paper we study the robust models of the Mean-Conditional Value at Risk (Mean-CVaR) portfolio selection problem under the estimation risk in mean return for both interval and ellipsoidal uncertainty sets. The corresponding robust models are a linear programming problem and a second order conic programming problem (SOCP) respectively. At the end an example is given to demonstrate the impact of uncertainty.

Keywords: Portfolio Optimization, Robust Optimization, Value at Risk, Conditional Value at Risk, Conic Optimization.

Authors

Farzaneh Piri, Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran. Email: farzanehpiri@yahoo.com
Maziar Salahi, Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran. Email: salahim@guilan.ac.ir
Farshid Mehrdoust, Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran. Email: fmehrdoust@guilan.ac.ir

Citation Farzaneh Piri, Maziar Salahi, Farshid Mehrdoust (2014). Robust Mean-Conditional Value at Risk Portfolio Optimization. *International Journal of Economic Sciences*, Vol. III, No. 1/2014, pp. 2-11

Introduction

The nature of the investment and business activities is such that to achieve return are required to bear the risk. Therefore, when deciding for the investment, an investor has to accept a balance between risk and return. Hence, portfolio optimization has been demonstrated as an important ploy in investment and has led to create many theories and models in this context [8].

One of the most famous theories is optimal selection of portfolio theory introduced by Harry Markowitz in 1952 [10]. The Markowitz mean-variance (MV) model has been used as the standard framework for optimal portfolio selection problems. In this model, portfolio return is represented by the expected return and risk of the portfolio is measured by the variance of the portfolio returns. The variance is a statistical dispersion measure that gives the average of the squared distance of the possible returns from the expected return. So, an asset with return better than expected return is assumed to be as risky as an asset with return lower than expected return, whereas most investors don't consider risk of the high return. Hence, the variance is an adequate measure for calculating risk only when the returns of the underlying assets are distributed symmetrically. Hence, financial institutions and individuals attention was attracted to risk management in terms of percentiles of loss distribution such as Value at Risk (VaR). Instead of regarding the whole spreading around the expected return, VaR considers only the spreading to the left of the expected return as risk and represents the predicted maximum loss with a specified confidence level (e.g. 95%) over a certain period of time (e.g. one day) [4,12].

VaR is widely used in the financial industry, but despite this popularity, it has undesirable properties that restrict its use [2, 9, 11]. One of these properties is that VaR lacks subadditivity (for a risk measure f, we should have $f(x_1 + x_2) \leq f(x_1) + f(x_2)$). Obviously diversification reduces risk, thus "The total risk of two different investment portfolios does not exceed the sum of the individual risks". Another drawback of VaR is that it is a non-convex function [11]. Thus, optimizing VaR in this manner might lead to local optimizers, while we look for the global minimum.

These drawbacks of VaR led to the development of alternative risk measures. One well-known modification of VaR is obtained by computing the expected loss given that the loss exceeds VaR. This quantity is called Conditional Value-at-Risk (CVaR). VaR implies that "what is the maximum loss that we realize?" but CVaR asks: "How do we expect to incur losses when situation is undesirable?" CVaR also considers only one tail of the loss distribution, which corresponds to high losses, and does not account for the opposite tail representing high returns. This is a preference over Markowitz approach that as was mentioned it considers both of the tail of the loss distribution. Numerical experiments show that minimum CVaR often lead to optimal solutions close to the minimum VaR, because VaR never exceeds CVaR. Thus, portfolios with low CVaR should also have a low VaR. In addition, when return (loss) distribution is normal, VaR and CVaR are equal namely they result same optimal portfolios. However for skewed return distributions, optimal VaR and CVaR portfolios may be quite different. In addition, the tail of the loss distribution in higher losses of VaR may be long, because using VaR, the higher losses of VaR are not controlled. But CVaR controls the higher losses of VaR, thereupon it is more conservative than VaR. Also CVaR optimization is a convex programming problem and thus it is easy to optimize [5].

In this paper, first we introduce Mean-CVaR model in Section 2. Then due to the unavoidable estimation error of the expected return of the assets, in Section 3 we utilize robust optimization to deal with it for interval and ellipsoidal uncertainty sets. The original model and the robust models are compared on an example showing some useful features of the latter ones.

1. Mean-Conditional Value at Risk

Consider assets $S_1,...,S_n$, $n \geq 2$, with random returns. Suppose μ_i and σ_i denote the expected return and the standard deviation of the return of asset S_i. Moreover ρ_{ij} let for $i \neq j$, denote the correlation coefficient of the returns of assets S_i and S_j, $\mu = [\mu_1, ..., \mu_n]$ and $Q = (\sigma_{ij})$ be the $n \times n$ symmetric covariance matrix. Now if we denote by x_i as the proportion of holding in the asset i, one can represent the expected return and the variance of the resulting portfolio x as follows:

$$E[x] = \mu_1 x_1 + ... + \mu_n x_n = \mu^T x,$$
$$Var(x) = \sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j x_i x_j = x^T Q x,$$

where $\rho_{ii} = 1$. Also, we will assume that the set of feasible portfolios is a nonempty polyhedral set and represent it as $\Omega = \{x \mid Ax = b, Cx \geq d\}$ where A is a $m \times n$ matrix, b is an m-dimensional vector, C is a $p \times n$ matrix and d is a p-dimensional vector. In particular, one of the constraints in the set Ω is $\sum_{i=1}^{n} x_i = 1$.

Let $f(x, y)$ denote the loss function when we choose the portfolio x from a set of feasible portfolios and y is the realization of the random events. We consider the portfolio return loss, $f(x, y)$, the negative of the portfolio return that is a convex (linear) function of the portfolio variables x:

$$f(x, y) = -y^T x = -[y_1 x_1 + ... + y_n x_n]. \qquad (1)$$

We assume that the random vector y has a probability density function denoted by $p(y)$. For a fixed decision vector x, the cumulative distribution function of the loss associated with that vector is computed as follows:

$$\psi(x, \gamma) = \int_{f(x,y)\leq\gamma} p(y)dy. \qquad (2)$$

Then, for a given confidence level α, the α-VaR associated with portfolio x is defined by

$$VaR_\alpha(x) = \min\{\gamma \in \mathbb{R} \mid \psi(x, \gamma) \geq \alpha\}. \qquad (3)$$

Moreover, we define the α-CVaR associated with portfolio x as follows:

$$CVaR_\alpha(x) = \frac{1}{(1-\alpha)} \int_{f(x,y)\geq VaR_\alpha(x)} f(x, y)\, p(y)dy. \quad (4)$$

Theorem 1-2. We always have: $CVaR_\alpha(x) \geq VaR_\alpha(x)$, that means CVaR of a portfolio is always at least as big as its VaR. Consequently, portfolios with small CVaR also have small VaR. However, in general minimizing CVaR and VaR are not equivalent.

Proof: See [5].

Since the definition of CVaR implies the VaR function clearly, it is difficult to work with and optimize this function. Instead, the following simpler auxiliary function is considered [5]:

$$F_\alpha(x,\gamma) = \gamma + \frac{1}{(1-\alpha)} \int_{f(x,y)\geq\gamma} (f(x,y)-\gamma)\, p(y)dy \quad (5)$$

or

$$F_\alpha(x,\gamma) = \gamma + \frac{1}{(1-\alpha)} \int (f(x,y)-\gamma)^+ p(y)dy \quad (6)$$

where $a^+ = \max\{a,0\}$. As a function of γ, F_α has the following properties that are useful in computing VaR and CVaR [5]:

1. F_α is a convex function of γ.

2. VaR_α is a minimizer over γ of F_α.

3. The minimum value over γ of the function F_α is $CVaR_\alpha$.

Using these properties, we have

$$\min_x CVaR_\alpha(x) = \min_{x,\gamma} F_\alpha(x,\gamma). \quad (7)$$

Thus, we can optimize CVaR directly, without needing to compute VaR first. Since we consider the loss function $f(x,y)$ is a convex (linear) function of the portfolio variables x, then $F_\alpha(x,\gamma)$ is also a convex (linear) function of x. Thus if the feasible portfolio set Ω is also convex, the optimization problems in (7) are convex optimization problems that are efficiently solvable.

Instead of using the density function $p(y)$ of the random events in formulation (6) which is often impossible or undesirable to compute, we can use a number of scenarios y_i, $i = 1,...,m$. Thus we consider the following approximation to the function $F_\alpha(x,\gamma)$:

$$\overline{F}_\alpha(x,\gamma) = \gamma + \frac{1}{(1-\alpha)m} \sum_{i=1}^{m} (f(x,y_i)-\gamma)^+ \quad (8)$$

Now (7) becomes

$$\min_{x,\gamma} \ \gamma + \frac{1}{(1-\alpha)m} \sum_{i=1}^{m} (f(x,y_i)-\gamma)^+. \quad (9)$$

This can be reformulated as follows:

$$\begin{aligned} \min_{x,z,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)m} \sum_{i=1}^{m} z_i \\ s.t. \quad & z_i \geq 0, \ \ i = 1,...,m \\ & z_i \geq f(x,y_i) - \gamma, \ \ i = 1,...,m \\ & x \in \Omega. \end{aligned} \quad (10)$$

which is a convex optimization problem for convex function f.

It should be noted that risk managers often try to optimize risk measure while expected return is more than a threshold value. In this case, the Mean-CVaR model is as follows:

$$\begin{aligned} \min_{x,z,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)m}\sum_{i=1}^{m} z_i \\ s.t. \quad & \mu^T x \geq R, \\ & z_i \geq 0, \quad i = 1,...,m \\ & z_i \geq f(x, y_i) - \gamma, \quad i = 1,...,m \\ & x \in \Omega. \end{aligned} \tag{11}$$

2. Robust Mean-CVaR model

The expected return of the assets as input parameter for the Mean-CVaR problem is not known, thus it should be estimated in order to solve the problem. But estimating it might lead to large error. The effect of the estimation error can be observed in Figure (1) that shows the resulting efficient frontiers of the Mean-CVaR problem with true and estimated parameters for an 8-asset example taken from [13]. It is observed from this figure that, the estimated efficient frontier is obviously below the true efficient frontier. Thereupon, for the same standard deviation, the expected return obtained by estimated efficient frontier is significantly lower. Also, when we focus more on maximizing expected return than minimizing risk, portfolios with less diversification are obtained. In this case, the estimation error can be more significant, especially when the estimated highest-return asset is different from the true one. So the maximum expected return portfolios can be relatively far away from each other. This is clear in Figure (1).

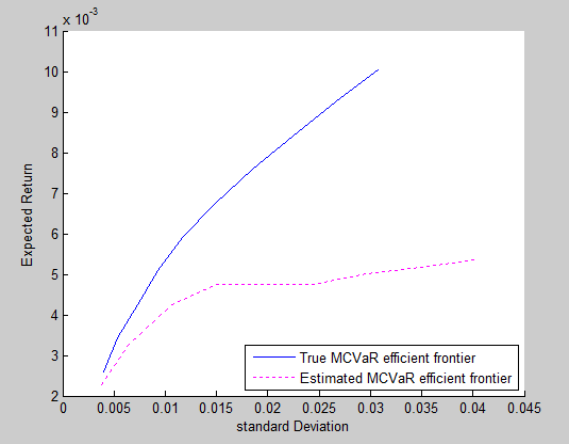

Figure 1. True efficient frontier and estimated efficient frontier

To deal with the estimation error, several methods such as robust optimization have been suggested [6, 7]. Robust optimization refers to the modeling of optimization problems with data uncertainty. Uncertainty in the parameters is described through uncertainty sets that contain all (or most) possible values that can be realized by the uncertain parameters. Uncertainty over the parameters is

discussable in several approaches. Two of the most common approaches are interval and ellipsoidal uncertainties. In this paper, we will use both of them and present equivalent models of robust Mean-CVaR model.

First let us consider Mean-CVaR model with the interval uncertainty over expected return, namely

$$\begin{aligned} \min_{x,z,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)m}\sum_{i=1}^{m} z_i \\ s.t. \quad & \mu^T x \geq R, \\ & z_i \geq 0, \quad i = 1,...,m \\ & z_i \geq f(x, y_i) - \gamma, \quad i = 1,...,m \\ & x \in \Omega, \\ & \mu \in S_\mu, \end{aligned} \tag{12}$$

where $S_\mu = \left\{\mu \mid \mu^L \leq \mu \leq \mu^U\right\}$, and also μ^L and μ^U are two given vectors. Since $\mu^T x \geq R$ should hold for all $\mu \in S_\mu$, thus (12) is equivalent to

$$\begin{aligned} \min_{x,z,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)m}\sum_{i=1}^{m} z_i \\ s.t. \quad & (\mu^L)^T x \geq R, \\ & z_i \geq 0, \quad i = 1,...,m \\ & z_i \geq f(x, y_i) - \gamma, \quad i = 1,...,m \\ & x \in \Omega. \end{aligned} \tag{13}$$

Obviously, if the function f is linear, then the problem is a linear programming problem, which is efficiently solvable using interior point methods.

However, if the uncertainty set is ellipsoidal, namely

$$S_\mu = \min\left\{\mu \mid \mu = \bar{\mu} + Mu, \ \| u \| \leq 1\right\},$$

where M is a n-dimensional matrix, then to establish $\mu^T x \geq R$ in the problem (11), we should have:

$$\begin{aligned} \min_{\mu} \mu^T x \geq R \ &\Leftrightarrow \ \min_{u} \left((\bar{\mu} + Mu)^T x\right) \geq R \\ &\Leftrightarrow \ (\bar{\mu})^T x + \min_{u} (Mu)^T x \geq R. \end{aligned}$$

Consequently it is enough to have

$$(\bar{\mu})^T x - \| M^T x \| \geq R$$

or $\| M^T x \| \leq (\bar{\mu})^T x - R$, which is a second order conic constraint. Thus, the robust Mean-CVaR problem (11) under ellipsoidal uncertainty set is as follows:

$$\begin{aligned} \min_{x,z,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)m} \sum_{i=1}^{m} z_i \\ s.t. \quad & \| M^T x \| \leq (\bar{\mu})^T x - R, \qquad (14) \\ & z_i \geq 0, \quad i = 1, ..., m \\ & z_i \geq f(x, y_i) - \gamma, \quad i = 1, ..., m \\ & x \in \Omega. \end{aligned}$$

Obviously, if the function f is linear, the problem (14) is a second order conic programming problem (SOCP) that is efficiently solvable using interior point methods [3].

3. Numerical Results

In this section, the performance of the robust Mean-CVaR models using interval and ellipsoidal uncertainty sets with Mean-CVaR model are compared using actual data. The dataset used here is available returns for eight assets related to 50 consecutive months that expected return and covariance matrix of the return of assets have been given in Tables (1) and (2).

The Mean-CVaR efficient frontier and the average of 100 frontiers obtained from two models are given in Figure (2). We observe that the efficient frontier for the interval uncertainty set lies above the efficient frontier obtained from the ellipsoidal uncertainty set.

This figure also indicates that resulting robust portfolios can be too conservative and maximum expected return obtained from them is significantly less than Mean-CVaR efficient frontier. Thus these methods can be appropriate for conservative investors to prevent losses due to initial parameter uncertainties, but it is not an appropriate choice for the investors who are more tolerant to estimation risk and wish to seek higher returns. However, the conservatism of the robust portfolios obtained using interval uncertainty set can be adjusted by decreasing the uncertainty interval. But it is important to note that the worst samples are eliminated in the new uncertainty sets and the robustness of the model is reduced compared with the robust frontier obtained from original interval uncertainty set in Figure (3), the frontiers obtained based on eliminating 2.5% and 5% of the worst samples μ in Figures (4) and (5) become longer and achieve higher maximum expected returns. It also shows that this model can be very sensitive to the choice of the boundary μ^L. Therefore, the uncertainty set must be carefully chosen.

Table 1. mean return vector μ

0.01×	S1	S2	S3	S4	S5	S6	S7	S8
	1.0160	0.4746	0.4756	0.4734	0.4742	-0.0500	-0.1120	0.0360

Table 2: covariance matrix Q

0.01×	S1	S2	S3	S4	S5	S6	S7	S8
S1	0.0980							
S2	0.0659	0.1549						
S3	0.0714	0.0911	0.2738					
S4	0.0105	0.0058	-0.0062	0.0097				
S5	0.0058	0.0379	-0.0116	0.0082	0.0461			
S6	-0.0236	-0.0260	0.0083	-0.0215	-0.0315	0.2691		
S7	-0.0164	0.0079	0.0059	-0.0003	0.0076	-0.0080	0.0925	
S8	0.0004	-0.0248	0.0077	-0.0026	-0.0304	0.0159	-0.0095	0.0245

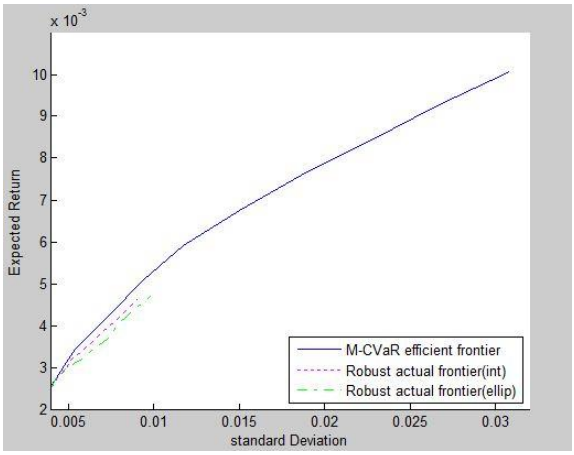

Figure 2. Robust Mean-CVaR frontiers using interval and ellipsoidal uncertainty sets

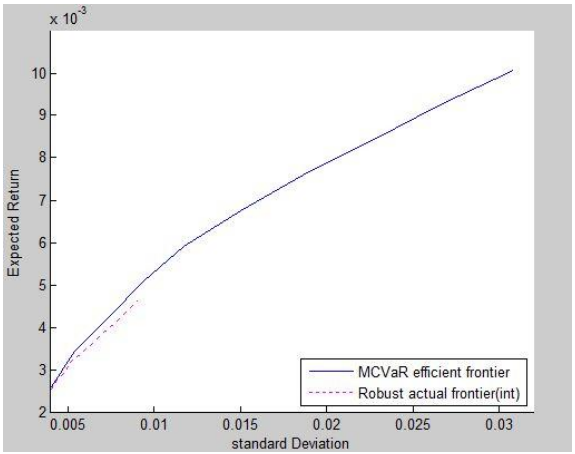

Figure 3. Robust Mean-CVaR frontiers using interval uncertainty set with eliminating 0% of the samples

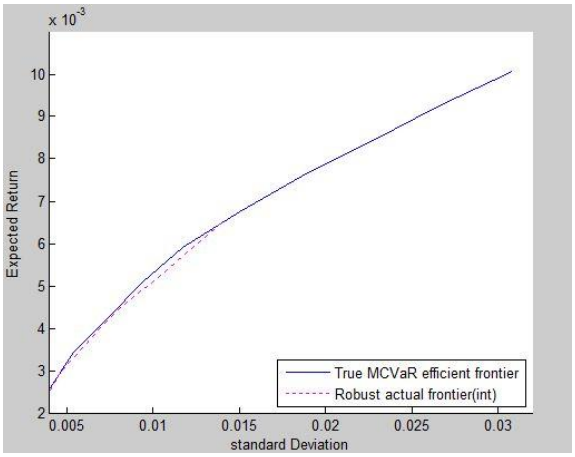

Figure 4. Robust Mean-CVaR frontiers using interval uncertainty set with eliminating 2.5% of the samples

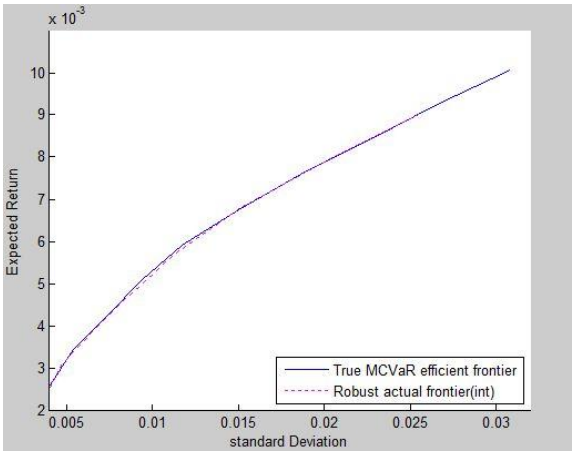

Figure 5. Robust Mean-CVaR frontiers using interval uncertainty set with eliminating 5% of the samples

4. Conclusions

Due to the estimation of the input parameters in the Mean-CVaR model, the performance of the Mean-CVaR efficient frontier can be very poor. In this paper, we focus on the estimation risk in the Mean-CVaR model. We consider estimation risk in the expected return of the assets and use interval and ellipsoidal uncertainty sets to represent robust Mean-CVaR models. We show that the robust portfolios can be too conservative and unable to obtain sufficiently high expected returns. Adjustment of the level of conservatism in the robust Mean-CVaR model using interval uncertainty set can be achieved by eliminating bad scenarios from the uncertainty set; but because of reducing the robustness of the model, this is unappealing. Therefore, the uncertainty set must be carefully chosen.

REFERENCES

[1] S. Alexander, T.F. Coleman and Y. Li, "Minimizing VaR and CVaR for a portfolio of derivatives," Journal of Banking and Finance, Vol. 30, No. 2, 2006, pp. 583–605.

[2] P. Artzner, F. Delbaen, J. Eber and D. Heath, "Thinking coherently, " Risk, Vol. 10, 1997, pp. 68–71.

[3] S. Boyd and M. Grant, "Cvx users, guide for cvx version 1.21," 2010. www.Stanford.edu/~boyd/cvx.

[4] W.N. Cho, "Robust Portfolio Optimization Using Conditional Value At Risk," Imperial College London, Department of Computing, Final Report, 2008.

[5] G. Cornuejols and R. Tutuncu, "Optimization Methods in Finance," Cambridge University Press, 2006.

[6] L. Garlappi, R. Uppal and T. Wang ,"Portfolio selection with parameter and model uncertainty: A multi-prior approach," Review of Financial Studies, Vol. 20, 2007, pp. 41–81.

[7] D. Goldfarb and G. Iyengar, "Robust portfolio selection problems," Mathematics of Operations Research, Vol. 28, No. 1, 2003, pp. 1–38.

[8] P. Krokhmal, J. Palmquist and S. Uryasev, "Portfolio optimization with conditional Value-at-Risk objective and constraints, " Journal of Risk, Vol. 4, 2002, pp. 43-48.

[9] M. Letmark, "Robustness of Conditional Value at Risk when Measuring Market Risk Across Different Asset Classes," Master Thesis, Royal Institute of Technology ,2010.

[10] H. Markowitz, "Portfolio selection," Journal of Finance, Vol. 7, 1952, pp. 77–91.

[11] A.G. Quaranta and A. Zaffaroni, "Robust Optimization of Conditional Value at Risk and Portfolio Selection," Journal of Banking and Finance, Vol. 32, 2008, pp. 2046–2056.

[12] R.T. Rockafellar and S. UryasevksiR fo lanruoJ ",risk-at-eulav lanoitidnoc fo noitazimitpO" ,, Vol. 2, No. 3, 2000, pp. 21–41.

[13] L. Zhu, T.F. Coleman and Y. Li, "Min-max robust and CVaR robust mean-variance portfolios," Journal of Risk, Vol. 11, No. 3, 2009, pp. 1-31.