JIE DAI

FISMS, Sobey School of Business, Saint Mary's University, Canada, jie.dai@smu.ca

ASSESSING SOLVENCY OF FINANCIAL INSTITUTIONS: AN OPTION-THEORETIC APPROACH

Abstract:

In this paper, we quantify the subtle concepts of soundness and safety of financial institutions through option pricing theory. This approach permits to intuit many solvency issues, including capital adequacy, financial strain or distress, balance sheet rebuilding and critical recapitalization, through private capital injections or government rescue programs. Numerical examples to implement the involved calculations are provided to illustrate the economic principles that underpin many governmentsponsored rescue plans, such as the Troubled Assets Relief Program (TARP) in the U.S. This option approach applies easily to the general topic of capital structure decisions and thereby improves upon the static theory which is usually limited to trade-off between tax benefits and bankruptcy costs.

Keywords:

Soundness and safety of financial institutions, Capital adequacy, Recapitalization

JEL Classification: G01, G28, G32

1. INTRODUCTION

The soundness and safety of financial institutions (FIs) are of primary concern for all countries. These institutions face various risks that affect their profits and value, including, foremost, that of insolvency and failure. Whenever regulators exert benevolent surveillance over FIs, the danger arises that competition and mimicking lead FIs to flout safety norms and engage in perilous lending, such as issuances of sub-prime mortgage. The turmoil in global capital markets triggered in 2007 by fragilized U.S. banks demonstrates the importance of maintaining a healthy and stable banking system in an economy.

Yet, in applied finance, we often lack a clearly defined concept of financial health and stability for a financial institution or for any business firm in general, and equally lack some readily quantifiable measure of the concept. In this paper, we fill this gap by proposing a general framework within which financial health and stability can be analyzed using the option pricing theory of firm valuation.[1]

All financial institutions, except mutual funds, have financial liabilities such as deposits for banks and claims for insurance companies, which represent the debt obligations of the FIs to their liability holders. When the asset value of an FI falls below its debt obligations, the FI loses all its net worth and faces technical insolvency, which may eventually drive the FI to its demise. While the factors determining the potential for collapse of an FI are inevitably many, the two key variables are quality of assets and amount of equity capital. Based on intuition from option pricing theory, both of these key variables can be captured in two formulas, namely, the *probability of insolvency* and the *risk capital* of a financial institution.

According to the seminal work by Black and Scholes [1973] and Merton [1974], a firm's equity mirrors a European call option on the firm's assets, due to the features of residual claim and limited liability of the equity position. Thus, by making the same assumptions that underlie the Black-Scholes-Merton option pricing model, we can calculate equity value as a call option on the firm's assets with a strike price equal to the liabilities of the firm.

Drawing on this option-theoretic approach to firm valuation, we develop a managerial tool that, based on two formulas measuring the FI's probability of insolvency and the level of its risk capital, can effectively cover, and connect with, many important factors affecting an entity's situation, such as financial leverage, potential for collapse, and feasibility of subsidy-free capital injection to achieve target financial health. This tool is easy to implement in practice but conveys much richer intuition than the typical treatment of capital structure used in finance analysis. Even in the context of policy discussions, this tool provides a solid framework within which some of the most current

[1] The option-based approach to firm valuation pioneered by Merton [1974] – now commonly referred to as "structural model" – is a widely popular framework for analyzing firm distress. Research adopting such an approach includes Ronn and Verma [1986], Vassalou and Xing [2004], Bharath and Shumway [2008], Schaefer and Strebulaev [2008], and Dai and LaPointe [2010], amongst others. In applications, many credit risk models of huge commercial successes like CreditMetrics and KMV are all variants of Merton's model. In regulations, the "Prompt Correction Actions" where regulators intervene to handle a frail FI are also guided by the economic principles from Merton's model.

and highly politicized debates can be addressed. For example, the framework can be used to clarify that the government bailout plan TARP is truly intended to help banks recapitalize and thereby stabilize the financial system, rather than to increase lending as such, or to subsidize banks.

In the following sections, we first lay out the theoretical foundations of the heuristic tool, with all the formulas derived and explained; we then present numerical examples to illustrate the implementation of the tool for three typical scenarios. To highlight, the following aspects will be covered:

a) Quantification of the elusive concepts of soundness and safety of financial institutions
b) Heuristic treatment of the controversial Troubled Assets Relief Program (TARP) initiated by The U.S. Treasury and Federal Reserve in the later part 2008
c) The option-theoretic approach to banking firms' capital structure and debt decisions
d) Calculations of capital adequacy and deleveraging
e) Calculations of the fair amount of capital infusion needed to acquire a certain stake in a troubled financial institution (such as what the TARP achieved in reverse auctions or what Buffett accomplished in investing in Goldman Sachs)
f) Identification of the subsidy-free capital injection to restore a target financial health
g) Use of Solver to find solutions to nonlinear equations stemming from the two formulas

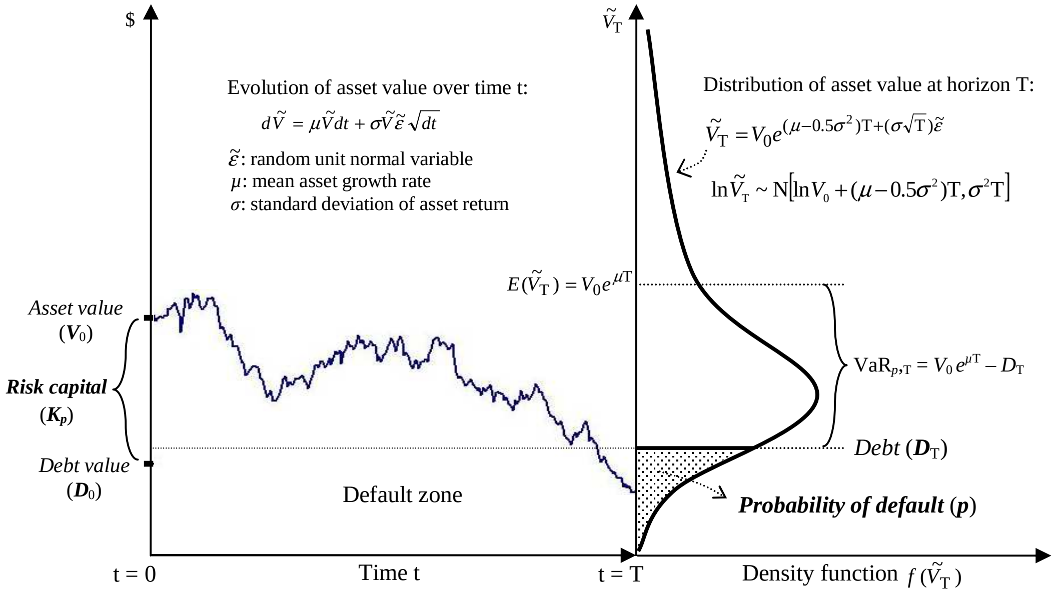
With the option approach adopted, we dispose of a practical tool with which to assess how the values of assets, debt, equity, and financial health *interrelate* broadly when FIs contemplate, or engage in, major decisions regarding their financial health. Indeed, no other financial theory and model has such ability to capture explicitly the interrelationships among these variables.

2. DEFAULT, RISK CAPITAL, AND OPTIONS: AN OVERVIEW

In this section, our objective is to develop a measure that can best indicate the financial health of a financial institution (FI). As will be seen, this development is built on several strands of financial literature, ranging from option pricing theory to its application in areas such as default, value at risk (VaR), and risk capital.

For ease of illustration, Figure 1 depicts the general framework within which our following analysis is cast. All the variables shown in the figure will be defined and explained in this section. Readers conversant with option pricing theory can skip to Sections 3 and 4 for applications and discussions.

Figure 1 – The General framework



Note: ***Risk capital*** is the minimum equity capital (K_p) required today to support business activities using assets of current value (V_0) so that ***probability of default*** on debt liability at horizon T (D_T) will be smaller than p.

Consider a financial institution (FI) whose assets are currently evaluated at V_0 and whose debt obligation at a later time T is D_T. Figure 1 shows that the event of default occurs when asset value ($\tilde{V}_T$) falls short of debt liability (D_T) at the debt's maturity (T). In the event of such a default, the FI loses all its net worth, equity, which may eventually drive the FI into bankruptcy and demise.

While factors determining the survival of an FI are inevitably many, the following ones are key: quality of assets; level of financial leverage; and amount of equity capital. Based on intuition from option pricing theory, we are able to capture all these important variables in just two formulas, namely, the *probability of insolvency* and the *risk capital* of a financial institution.

- *Probability of insolvency*:

$$p = N\left(-\frac{\ln(V_0/D_T) + (\mu - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \quad \text{---- (a)}$$

In general, for an financial institution, the better (poorer) the quality of its assets, as measured by the ratio of return to risk, μ/σ, and the lower (higher) its financial leverage, as measured by the ratio of debt to asset, D_T/V_0, the smaller (larger) the chance of insolvency, p.

- *Risk capital*:

$$K_p = V_0 \mathrm{N}(d_1) - D_\mathrm{T} e^{-r_f \mathrm{T}} \mathrm{N}(d_2) \qquad \text{---- (b)}$$

where $$d_1 = \frac{\ln\left(V_0 / D_\mathrm{T}\right) + (r_f + 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$$

$$d_2 = \frac{\ln\left(V_0 / D_\mathrm{T}\right) + (r_f - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}} = d_1 - \sigma\sqrt{\mathrm{T}}\,.$$

For a financial institution, the important issue of capital adequacy is resolved by examining whether its current equity is larger or not than the risk capital calculated at a confidence level which is set equal to the probability of insolvency as shown in equation (a).

In the next, we will illustrate the derivations of the above two formulas for default probability, (a), and risk capital, (b). We aim to provide complete and heuristic derivations so that a reader will have a handy reference for all the related technical aspects of the two formulas.

To start, the following economic and technical assumptions are used:

1) The market is perfect, with no transaction costs and no taxes.
2) The Modigliani-Miller theorem holds: that is, the value of the business assets (*V*) is invariant to how the business is financed, i.e., independent of capital structure – the mix of equity capital and debt liability, $V_t = E_t + D_t$.
3) The business's debt pays zero coupons; also, the equity pays no dividends between time 0 and debt maturity T.[2]
4) The value of the business' assets, *V*, follows a *geometric Brownian motion* (GBM) process:

$$dV = \mu V dt + \sigma V dZ \qquad \text{---- (1)}$$

with $$dZ = \varepsilon\sqrt{dt} \qquad \text{---- (2)}$$

where ε is a random variable following standard/unit Normal distribution.

Equation (1) defines the stochastic process of V i.e., it describes how asset value evolves over time. Here, μ is the instantaneous growth rate of the asset value per unit time, and σ is the instantaneous volatility of the growth rate per unit time. To intuit geometric Brownian motion, we can think the percentage change in asset value, dV/V, during an infinitely short time period, dt, as being caused by two factors: one is a deterministic increase, μdt, which is determined by growth rate, and the other is a random shock, σdZ, which is determined by volatility.

[2] The no payout assumptions here are merely for focusing on the main idea; see Merton [1974] for how to incorporate dividend payments.

Equation (2) defining dZ is a Wiener process which tells how the random shocks are generated as time passes by. To intuit this shock generating process, we can think that after every infinitely short time period, dt, a value for ε is drawn from a standard normal distribution. The product of this ε and $\sqrt{dt}$ is dZ, which gives rise to randomness in dV. The Wiener process is a basic building block for construction of many other more sophisticated stochastic processes.[3]

Given the stochastic process of asset value, equation (1), it can be shown, with the help of a mathematical rule from stochastic calculus called Itô's Lemma, that asset value at time T, $\tilde{V}_{\mathrm{T}}$, is uncertain and follows a lognormal distribution.[4] That is, the logarithm of asset value at horizon time T, $\ln\tilde{V}_{\mathrm{T}}$, is normally distributed with the *expected value*

$$E(\ln\tilde{V}_{\mathrm{T}}) = \ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T} \qquad \text{---- (3)}$$

and *variance*

$$Var(\ln\tilde{V}_{\mathrm{T}}) = \sigma^2\mathrm{T}. \qquad \text{---- (4)}$$

Often we use the expression random variable ~ N [*Mean*, *Variance*] to indicate that the random variable follows a normal distribution. Thus, we can express $\ln\tilde{V}_{\mathrm{T}}$ as

$$\ln\tilde{V}_{\mathrm{T}} \sim \mathrm{N}\left[\ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T}, \sigma^2\mathrm{T}\right]. \qquad \text{---- (5)}$$

Based on results (3), (4) and (5), it is easy to derive the probability of default (p) at time T:

$$\begin{aligned} p\,(default) &= p\,(\tilde{V}_{\mathrm{T}} < D_{\mathrm{T}}) \\ &= p\,(\ln\tilde{V}_{\mathrm{T}} < \ln D_{\mathrm{T}}) \\ &= p\left(\frac{\ln\tilde{V}_{\mathrm{T}} - [\ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T}]}{\sigma\sqrt{\mathrm{T}}} < \frac{\ln D_{\mathrm{T}} - [\ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T}]}{\sigma\sqrt{\mathrm{T}}}\right) \\ &= \mathrm{N}\left(\frac{\ln D_{\mathrm{T}} - [\ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T}]}{\sigma\sqrt{\mathrm{T}}}\right) \\ &= \mathrm{N}\left(-\frac{\ln(V_0/D_{\mathrm{T}}) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}\right) = \mathrm{N}\,(-\hat{d}_2). \end{aligned}$$

Note that in the third equality, we have deducted from the normal variable $\ln\tilde{V}_{\mathrm{T}}$ its expected value, $\ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T}$, and then divided the difference by the standard

[3] According to the Wiener process, the cumulative sum of all the shocks over the time period from 0 to T, ($Z_{t=T} - Z_{t=0}$), is normally distributed with mean $E(Z_{t=T} - Z_{t=0}) = 0$, and variance $Var(Z_{t=T} - Z_{t=0}) = \mathrm{T}$, where Z_t is the value of the Wiener process at time point t.

[4] See Appendix for GBM, Itô's Lemma, and lognormal distribution. The tilde '~' designates a random variable.

deviation of $\ln \tilde{V}_{\mathrm{T}}$, $\sigma\sqrt{\mathrm{T}}$. Essentially, we have *standardized* the normal variable $\ln \tilde{V}_{\mathrm{T}}$, converting it to a unit normal variable with zero mean and unit variance, which is why we have changed p to N in the fourth equality.[5]

Thus, *Probability of insolvency* is expressed by

$$p\ (default) = \mathrm{N}\left(-\frac{\ln\left(V_0/D_{\mathrm{T}}\right) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\ \sqrt{\mathrm{T}}} \right) = \mathrm{N}\ (-\hat{d}_2) \qquad \text{---- (6)}$$

where N is the cumulative standard/unit Normal distribution function, and

$$\hat{d}_2 = \frac{\ln\left(V_0/D_{\mathrm{T}}\right) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$$

$$= \frac{[\ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T}] - \ln D_{\mathrm{T}}}{\sigma\sqrt{\mathrm{T}}} = \frac{E(\ln \tilde{V}_{\mathrm{T}}) - \ln D_{\mathrm{T}}}{\sigma\sqrt{\mathrm{T}}}. \qquad \text{---- (6.1)}$$

$\hat{d}_2$ is the difference between the expected value of $\ln \tilde{V}_{\mathrm{T}}$ and the default point lnD_{T}, divided (standardized) by standard deviation, $\sigma\sqrt{\mathrm{T}}$. Thus, $\hat{d}_2$ measures *how many standard deviations* the expected value of $\ln \tilde{V}_{\mathrm{T}}$ is away from the default point lnD_{T}. $\hat{d}_2$ is often called *standardized distance to default* (standardized DTD).

Now we turn to risk capital, which is built on value at risk, or "VaR". VaR is a probabilistic measure of potential loss to an existing wealth position, formally defined as the worst expected loss with some selected confidence level over a certain time period. It is the gap between the expected value and the value that will not be exceeded with a probability p (or confidence level 1 - p), i.e., the p^{th} quantile of the possible values of a position.[6] If we set this p^{th} quantile just equal to the debt liability at horizon T, D_{T}:

$$p^{\mathrm{th}}\ \text{quantile} = D_{\mathrm{T}} \qquad \text{---- (7)}$$

then we can connect the default probability, p, with asset value and debt liability at time T, $\tilde{V}_{\mathrm{T}}$ and D_{T}, in the following way: with probability p, asset value $\tilde{V}_{\mathrm{T}}$ will be smaller than debt liability D_{T}.

The logarithm of asset value at horizon time T, $\ln \tilde{V}_{\mathrm{T}}$, is normally distributed with mean $\ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T}$ and standard deviation $\sigma\sqrt{\mathrm{T}}$. With a normal distribution, we

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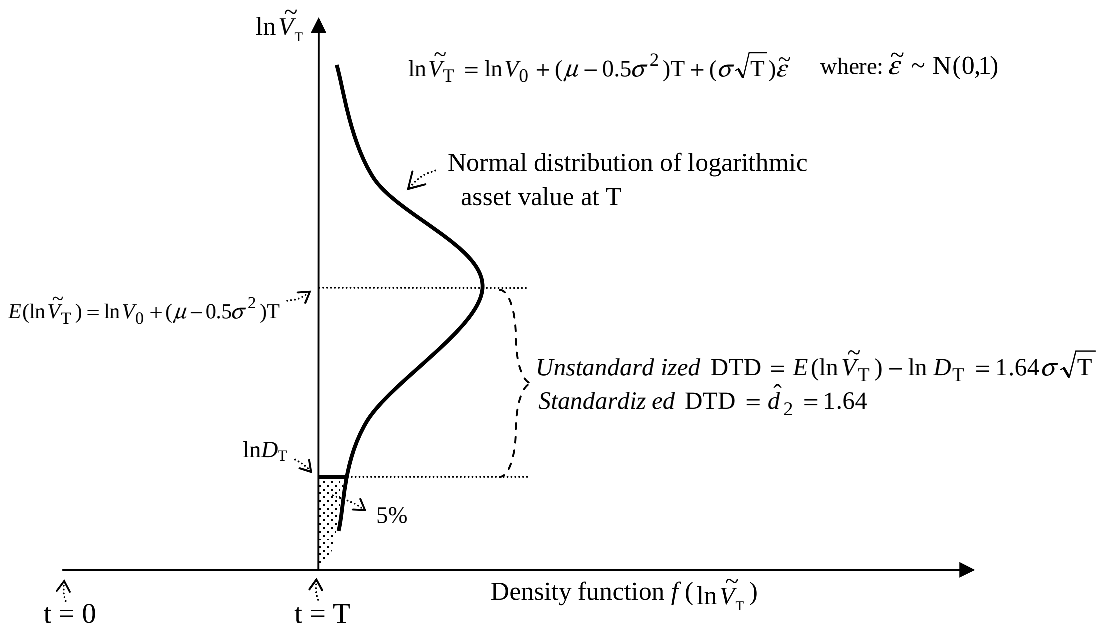
[5] We can always convert an arbitrary normal variable $\tilde{n} \sim \mathrm{N}\ (\mu, \sigma^2)$ to a unit normal variable $\tilde{\varepsilon} \sim \mathrm{N}\ (0, 1)$. This is done by first deducting the mean and then dividing by the standard deviation: $\tilde{\varepsilon} = (\tilde{n} - \mu)/\sigma$ -- a process called standardization of a normal variable. Conversely, we can also reverse the process, generating an arbitrary normal variable from a unit normal variable by multiplying the standard deviation and then adding the mean: $\tilde{n} = \tilde{\varepsilon}\sigma + \mu$.

[6] See the Appendix for the p^{th} quantile of a lognormal distribution, which differs from the p^{th} quantile of a normal distribution by the scale of "ln", i.e., ln (p^{th} quantile of a lognormal) = p^{th} quantile of a normal.

know that the 95%[th] percentile of the normal variable $\ln \tilde{V}_T$ is 1.64 standard deviations away from the expected value of $\ln \tilde{V}_T$. That is, on the "ln" scale, the particular $\ln \tilde{V}_T$ that is $1.64\sigma\sqrt{T}$ below the expected value of $\ln V_T$, gives the probability of 5%. If this particular $\ln \tilde{V}_T$ happens to be $\ln D_T$, then 5% is the default probability, as shown in Figure 2, where the unstandardized distance to default (DTD) is $E(\ln \tilde{V}_T) - \ln D_T = 1.64\sigma\sqrt{T}$, while the standardized distance to default $\hat{d}_2$ = 1.64.

Figure 2 – Distance to Default (DTD) at maturity T



Now that $\ln \tilde{V}_T \sim N[\ln V_0 + (\mu - 0.5\sigma^2)T, \sigma^2 T]$, that is, $\ln \tilde{V}_T$ follows a normal distribution with mean $\ln V_0 + (\mu - 0.5\sigma^2)T$ and standard deviation $\sigma^2 T$, we can standardize the normal variable $\ln \tilde{V}_T$ by first deducting its mean and then dividing the remainder by the standard deviation, i.e., we convert the normal variable $\ln \tilde{V}_T$ into a unit normal variable $\tilde{\varepsilon}$:

$$\tilde{\varepsilon} = \frac{\ln \tilde{V}_T - Mean}{\sqrt{Variance}} = \frac{\ln \tilde{V}_T - [\ln V_0 + (\mu - 0.5\sigma^2)]}{\sigma\sqrt{T}}.$$

After rearranging, we get

$$\ln \tilde{V}_T = \ln V_0 + (\mu - 0.5\sigma^2)T + \sigma\sqrt{T}\tilde{\varepsilon} \quad \text{---- (8)}$$

where $\tilde{\varepsilon} \sim N(0, 1)$.

When we apply exponentiation for the above equation, it becomes

$$\tilde{V}_{\mathrm{T}} = V_0 \times e^{(\mu - 0.5\sigma^2)\mathrm{T}} \times \left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right) \qquad \text{---- (9)}$$

By take expectation of the above variables using normal distribution, we get

$$E(\tilde{V}_{\mathrm{T}}) = V_0 \times e^{(\mu - 0.5\sigma^2)\mathrm{T}} \times E\left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right). \qquad \text{---- (10)}$$

We now calculate the part $E\left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right)$ in equation (10).

By definition of expected value:

$$E\left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right) = \int_{-\infty}^{\infty} e^{\sigma\sqrt{\mathrm{T}}\varepsilon} f(\varepsilon) d\varepsilon$$

where $f(\varepsilon)$ is the probability density function of unit normal variable $\tilde{\varepsilon}$.

The next few lines illustrate the calculations for $E\left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right)$.

With $\varepsilon \sim$ N (0, 1), $f(\varepsilon) = \frac{1}{1 \times \sqrt{2\pi}} e^{-0.5\left(\frac{\varepsilon - 0}{1}\right)^2}$,

$$\begin{aligned}
E\left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right) &= \int_{-\infty}^{\infty} e^{\sigma\sqrt{\mathrm{T}}\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-0.5\varepsilon^2} d\varepsilon \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(\sigma\sqrt{\mathrm{T}}\varepsilon - 0.5\varepsilon^2)} d\varepsilon \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5(\varepsilon^2 - 2\sigma\sqrt{\mathrm{T}}\varepsilon)} d\varepsilon \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5(\varepsilon^2 - 2\sigma\sqrt{\mathrm{T}}\varepsilon + \sigma^2\mathrm{T} - \sigma^2\mathrm{T})} d\varepsilon \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5(\varepsilon^2 - 2\sigma\sqrt{\mathrm{T}}\varepsilon + \sigma^2\mathrm{T}) + (0.5\sigma^2\mathrm{T})} d\varepsilon \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5(\varepsilon - \sigma\sqrt{\mathrm{T}})^2} e^{0.5\sigma^2\mathrm{T}} d\varepsilon \\
&= e^{0.5\sigma^2\mathrm{T}} \int_{-\infty}^{\infty} \frac{1}{1 \times \sqrt{2\pi}} e^{-0.5\left(\frac{\varepsilon - \sigma\sqrt{\mathrm{T}}}{1}\right)^2} d\varepsilon \\
&= e^{0.5\sigma^2\mathrm{T}}
\end{aligned}$$

The last equality follows because the integral is the total area beneath a normal density function with the mean of $\sigma\sqrt{\mathrm{T}}$ and the variance of unity. Therefore, the integral is one. Thus, we have established that

$$E\left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right) = e^{0.5\sigma^2\mathrm{T}}. \qquad \text{---- (11)}$$

Substituting equation (11) for expression of $E(\tilde{V}_\mathrm{T})$ in equation (10), $E(\tilde{V}_\mathrm{T})$ becomes

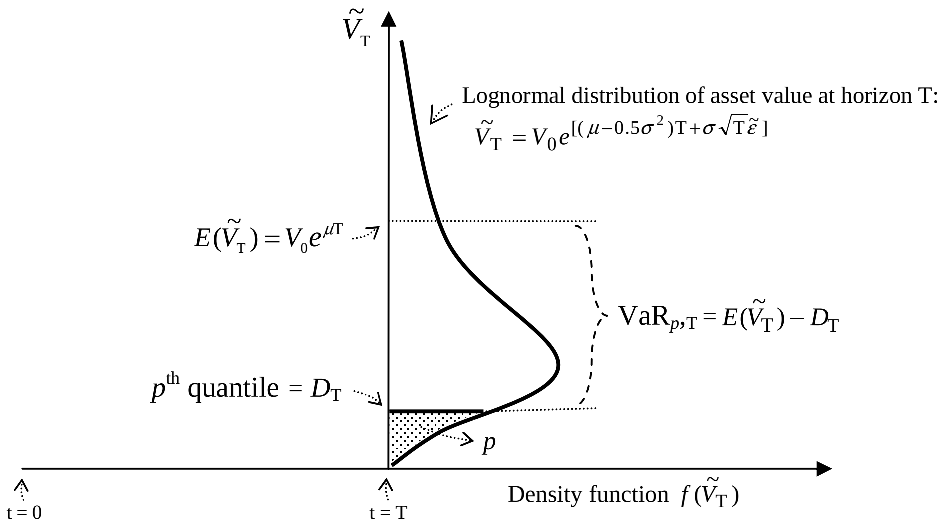
$$\begin{aligned} E(\tilde{V}_\mathrm{T}) &= V_0 \times e^{(\mu - 0.5\sigma^2)\mathrm{T}} \times E\left(e^{\sigma\sqrt{\mathrm{T}}\tilde{\varepsilon}}\right) \\ &= V_0 \times e^{(\mu - 0.5\sigma^2)\mathrm{T}} \times e^{0.5\sigma^2\mathrm{T}} \\ &= V_0 \times e^{[(\mu - 0.5\sigma^2)\mathrm{T} + 0.5\sigma^2\mathrm{T}]} \\ &= V_0 \times e^{\mu\mathrm{T}} \end{aligned}$$

Thus,

$$E(\tilde{V}_\mathrm{T}) = V_0 e^{\mu\mathrm{T}}. \qquad \text{---- (12)}$$

Now we have converted the various quantities and variables measured in the “ln” scale back to the “direct” scale, as illustrated in Figure 3:

Figure 3 – Lognormal distribution of asset value at maturity T



Thus, we get

$$\begin{aligned} \mathrm{VaR}_{p,\mathrm{T}} &= E(\tilde{V}_\mathrm{T}) - p^{\mathrm{th}} \text{ quantile} \\ &= V_0 e^{\mu\mathrm{T}} - D_\mathrm{T} \end{aligned} \qquad \text{---- (13)}$$

Note that this value for $\mathrm{VaR}_{p,\mathrm{T}}$ in equation (13) is measured at horizon time T, not present time 0. However, we want to know what minimum equity capital should be prepared today at time 0 so that the default probability at horizon time T can be kept below p. Thus, we need to convert the $\mathrm{VaR}_{p,\mathrm{T}}$ at horizon T to present time 0 to come up with risk capital K_0. This conversion from T to 0 relies, however, on insight gained from option pricing theory, rather than on a simple discounting process (although continuous discounting at a risk-free interest rate, $e^{-r_f \mathrm{T}}$, may give an approximation in the conversion).

The minimum equity capital – *risk capital* K_p – required today to support business activities/assets of current value (V_0) so that probability of default on debt liability (D_T) at horizon T will be smaller than *p* is given by the *Black-Scholes option pricing formula*. Thus, *Risk capital* can be expressed as

$$K_p = V_0 \mathrm{N}(d_1) - D_\mathrm{T} e^{-r_f \mathrm{T}} \mathrm{N}(d_2) \qquad \text{---- (14)}$$

where
$$d_1 = \frac{\ln\left(V_0 / D_\mathrm{T}\right) + (r_f + 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}} \qquad \text{---- (14.1)}$$

$$d_2 = \frac{\ln\left(V_0 / D_\mathrm{T}\right) + (r_f - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}} = d_1 - \sigma\sqrt{\mathrm{T}}\,. \qquad \text{---- (14.2)}$$

r_f is the risk-free interest rate, and N is the cumulative standard normal distribution function.[7]

Note that the variable d_2 in equation (14.2) of the Black-Scholes option pricing formula is the same as the distance to default $\hat{d}_2$ in equation (6.1) of the default probability formula, except that d_2 uses *risk-free* interest rate r_f, whereas $\hat{d}_2$ uses the *true* rate of return μ. d_2 and $\hat{d}_2$ differ only by this rate of return.[8] For pricing purposes, as with the Black-Scholes option pricing formula, a risk-free interest rate, r_f, is used, but when measuring the physical probability of default, the actual rate of return, μ, should be used in the default probability formula.

3. SCENARIOS OF HEALTHY, STRAINED, AND RECAPITALIZED INSTITUTIONS

Since the lending crisis erupted in 2007, the solvency of banks has become a major and unending concern in the USA first, before it spread to Europe. Central banks favored interventions to keep numberless banks afloat, whilst conservative financial economists, such as Fama and French (2009), kept advising, in vain, that frail banks should be left to their lot.

In this section, we employ numerical examples to illustrate how formulas (6) for default probability and (14) for risk capital or equity cushion can be used to address

[7] It is the close *analogy* between *call option* and *levered equity* that allows us to directly use the Black-Scholes call option pricing formula for risk capital. This analogy was first established in the seminal work by Merton [Merton, 1974].

[8] A key and apparently surprising insight from the Black-Scholes option pricing theory is that the pricing of an option is done through risk-neutral valuation, i.e., *risk aversion plays no role in pricing an option* (*but does play role in pricing the underlying asset*). This is so because option pricing is based on the *absence of risk-free arbitrage* opportunities *over any time interval*, and within this framework, only the time value of money (i.e., risk-free interest rate) is relevant when it comes to rate of return earned over time. Thus, the appropriate rate of return in option pricing is the risk-free interest rate, r_f, rather than the true physical rate of return, μ□

important issues faced by destabilized financial institutions (FIs). Let's consider the spectrum of three representative health scenarios for the same FI, going from healthy, to strained, to recapitalized.

3.1 The Healthy Financial Institution

Consider an FI with a strong commitment to keeping its default probability *p* below 1%. Assume its present asset value $V_0 = \$300$ million with an expected continuous asset growth rate per annum $\mu = 8\%$ and corresponding asset return volatility $\sigma = 15\%$. The risk-free interest rate stands at *r* = 5% over horizon T = 1 year. What minimum equity capital today will be consistent with a debt level D_{T} and default probability *p* = 1% one year ahead?

First, we use formula (6) in inverse fashion to find the implied debt liability, D_{T}:

$$\text{NORMINV}(p, 0, 1) = \left(-\frac{\ln(V_0/D_{\mathrm{T}}) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}} \right) \qquad \text{---- (15)}$$

We know that for *p* = 1%, NORMINV (1%, 0, 1) = **–**2.33. (For other arbitrary probabilities, this inversion process can be performed by using Excel or referring to a z-table).

Thus, $-2.33 = \left(-\frac{\ln(V_0/D_{\mathrm{T}}) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}} \right)$

Solving, D_{T} = $226.69 million

This D_{T} is the *promised value* of debt at time T, which *implies* a 1% bankruptcy probability for a healthy financial institution.

Second, we use formula (14) to compute the risk capital:

$$K_p = V_0\mathrm{N}(d_1) - D_{\mathrm{T}}e^{-r_f\mathrm{T}}\mathrm{N}(d_2)$$

$$= 300\mathrm{N}(2.28) - 226.69e^{-r_f\mathrm{T}}\mathrm{N}(2.13) = \$84.55 \text{ million}$$

where $d_1 = \frac{\ln(V_0/D_{\mathrm{T}}) + (r_f + 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}} = 2.28$ and $d_2 = d_1 - \sigma\sqrt{\mathrm{T}} = 2.13$.

This K_p of $84.55 is the minimum equity needed at time 0 to support asset V_0 = $300, debt liability D_{T} = $226.69 a year from now, and bankruptcy probability *p* = 1%.

Finally, since B/S identity always holds, we have:

$$V_0 = D_0 + E_0 = D_0 + K_p \qquad \text{---- (16)}$$

thus $D_0 = V_0 - K_p = 300 - 84.55 = \$215.45$ million

This is the *market value* of debt at time 0, when $E_0 = K_p$ = $84.55 and *p* = 1%.

Here we have the situation of a healthy firm with ample equity $(K_p = 84.55)$ and default probability *p* = 1%, given its asset value of $V_0 = 300$ and total debt due in 1 year $D_1 = 226.69$. Of course, excess equity $(> K_p)$ may be desirable, especially when bumpy times loom ahead. Otherwise the excess would best be used, for instance, by increasing dividends directly or via share repurchases. Note that because debt is almost default-free here (*p* = 1%), its discounted value is very nearly equal to its current market value $[226.69e^{-0.05} = 215.63 \approx D_0 = 215.45]$. We will see next that the gap widens as default probability *p* increases.

3.2 The Strained Financial Institution

Suppose the FI's asset value falls by 20% to $V_0' = \$240$ miilion, following a shock, say important mortgage write-offs, whilst its redeemable debt at T = 1 remains at \$226.69 million. What are its new default probability and balance sheet components?

First, we use formula (6) to directly determine the risk of insolvency as:

$$p = \mathrm{N}\left(-\frac{\ln\left(V_0'/D_\mathrm{T}\right) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\ \sqrt{\mathrm{T}}}\right) = \mathrm{N}(-0.84) = 20.08\%.$$

Second, we use the valuation formula for risky debt,

$$D_0' = D_\mathrm{T} e^{-r_f \mathrm{T}}\left[\frac{\mathrm{N}(h_1)}{d} + \mathrm{N}(h_2)\right] \qquad \text{---- (17)}$$

where

$$h_1 = -\frac{0.5\sigma^2\mathrm{T} - \ln(d)}{\sigma\sqrt{\mathrm{T}}}, \quad h_2 = -\frac{0.5\sigma^2\mathrm{T} + \ln(d)}{\sigma\sqrt{\mathrm{T}}} = -\left(h_1 + \sigma\sqrt{\mathrm{T}}\right), \quad \text{and} \quad d = \frac{D_\mathrm{T}}{V_0'} e^{-r_f \mathrm{T}}$$

to calculate the new market value of debt at time 0, D_0':

D_0' = \$210.88 million (< \$215.45 million; a decrease of \$4.57 million).

Since B/S identity always holds $V_0' = D_0' + E_0'$, 240 = 210.88 + E_0'. Thus,

E_0' = \$29.12 million (< \$84.55 million; a decrease of \$55.43 million).

Note that the drop of 60, from 300 to 240, in present asset value, affects foremost the prime risk takers, the stockholders of the FI. They lose $E_0 - E_0' = 84.55 - 29.12 = 55.43$. The loss remaining, 4.57, is borne by debtholders whose stake value declines from $D_0 = 215.45$ to $D_0' = 210.88$, since debt becomes more risky when underlying asset value falls. So every time V_0 changes, E_0 and D_0 change. This realistic *dual* effect makes the option approach to solvency issues more attractive than classic static theory which ascribes asset value changes *solely* to stockholders.

Note also that the asset value drop has strained the FI's situation. It has meant a twentyfold jump in default probability *p* (from 1% to 20%) and an equity loss of 55.43 on 84.55, or 66%. Finally, should the intent be to restore *p* to 1%, a recovery of 55.43 in equity value will suffice, since debt value would gain 4.57 in the process. Thus with the dual effect on E_0 and D_0 totaling 60.00, asset value would "return" to 300 from its V_0' level of 240.

This numerical example illustrates how the important concepts of equity, debt, and financial health are determined simultaneously. In fact, no other financial model can capture such explicit interrelationships among these most important variables for a business entity.

3.3 Recapitalizing a Strained Financial Institution

When a private investor steps in to strengthen a strained FI by injecting capital *C* in equity, he ensures that his stake in the resulting equity is worth at least *C*. Should the government be always able to condition its bailouts on its getting stakes equal to the funds provided, then no taxpayers' money would be squandered. But then, just like the private investor would do, it would only salvage entities with sizeable asset values, as is shown further in the text.

Now consider a case of capital injection by an investor with majority ownership in mind. What will capital injection C be if the investor seeks majority ownership (say 51%), as in the sketched balance sheet at right in Figure 4? And what will be its effect on default probability *p*?

Note first that injecting cash C can only but raise asset value by the same amount, thus the post-injection asset value $V_0^{''} = (240 + C)$. As for the boosted equity $(E_0^{''} + C)$, it is worth the call on revalued assets $V_0^{''}$, conditional on debt repayment D_{T}. So, inserting $D_{\mathrm{T}} = 226.69,\ r = 0.05,\ \sigma = 0.15,\ \mathrm{T} = 1$ and $V_0^{''} = (240 + C)$ in call formula (14), we have:

$$E_0^{''} + C = (240 + C)\mathrm{N}(d_1) - D_{\mathrm{T}} e^{-r\mathrm{T}} \mathrm{N}(d_2) \qquad \text{---- (18)}$$

where $d_1 = \frac{\ln(V_0^{''}/D_{\mathrm{T}}) + (r + 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}} = \frac{\ln[(240+C)/226.69] + [0.05 + 0.5(0.15)^2]1}{0.15\sqrt{1}}$, and

$$d_2 = d_1 - \sigma\sqrt{\mathrm{T}} = d_1 - \left(0.15\sqrt{1}\right). \qquad \text{---- (19)}$$

If the recapitalization will lead to a majority stake in the ownership, then

$$\frac{C}{E_0^{''} + C} = 51\%. \qquad \text{---- (20)}$$

Thus, $E_0^{''} + C = \frac{C}{0.51}$. Substituting this expression in equation (18), we get

$$C/0.51 = (240 + C)\mathrm{N}(d_1) - 226.69 e^{-0.05(1)} \mathrm{N}(d_2) \qquad \text{---- (21)}$$

This is a non-linear equation and does not have a closed-form solution for C. In such a case, we can proceed by trial-and-error. The right-hand side (RHS) will equal *C*/0.51 for the unique *C*, say C^*, that constitutes the solution. Let's try *C* = 26. Then *C*/0.51 = 50.98 and the RHS = 51.64. The RHS differ positively by θ = 0.66. Increasing *C* to 27, we get *C*/0.51 = 52.94 whereas RHS = 52.57, the difference being negative (θ = -0.37). Thus, C^* lies in the short range between 26 and 27. With an in-between solution, as is the case here, we can get an excellent approximation by linear interpolation (versus an extrapolation when the solution lies closely outside the range tested). The method obeys the property of two parallel right triangles whereby the ratio of bases [here in terms of C differences: $(C^* - 26)/(27 - 26)$] equals the ratio of heights [here in terms of θ differences: $(0.66 - 0.00)/(0.66 + 0.37) = 0.64$]. So we can write: $C^* = 26 + (1.00)(0.64) = 26.64$.

Alternatively, we can use Solver in Excel to quickly find the value for $C^* = \$26.68$ (see the appended Excel spreadsheet for illustrated calculations).

Using the unique $C = C^* = 26.68$ in the RHS of equation (18) yields the boosted equity value, that is, given the probabilities involved:[9]

$E_0^{''} + C^* = (240 + 26.68)\mathrm{N}(d_1) - 226.69e^{-0.05(1)}\mathrm{N}(d_2) = 52.32.$

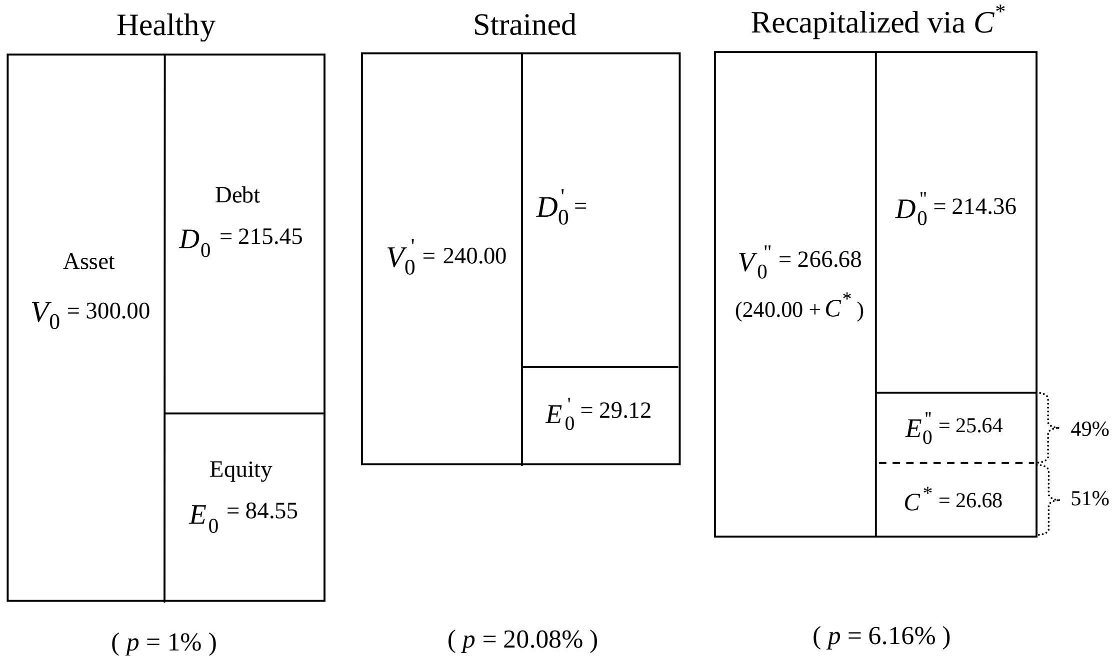
Thus, the original shareholders' equity value, post-recapitalization, is:

$E_0^{''} = 52.32 - 26.68 = 25.64,$

which implies a decline of 3.48, compared to its pre-recapitalization level $E_0^{'} = 29.12$. This decline is a wealth transfer from shareholders to debtholders, since the pre-recapitalization debt value $D_0^{'} = 210.88$ has increased by 3.48 to the post-recapitalization level $D_0^{''} = 214.36$. As for default probability p, it dropped from 20.08% to 6.16%, since with $V_0^{''} = 266.68$, $D_\mathrm{T} = 226.69$, $\mu = 0.08$, $\sigma = 0.15$, and $\mathrm{T} = 1$, using equation (6), we can ascertain that $p = \mathrm{N}\left(-\hat{d}_2\right) = \mathrm{N}\{-[\ln(V_0^{''}/D_\mathrm{T}) + (\mu - 0.5\sigma^2)\mathrm{T}]/\sigma\sqrt{\mathrm{T}}\} = \mathrm{N}(-1.5417) = 6.16\%$.

[9] $d_1 = [\ln(266.68/226.69) + (0.05 + 0.5(0.15)^2)1]/(0.15*1) = 1.4915$; $\mathrm{N}(d_1) = 0.9321$; $d_2 = d_1 - \sigma\sqrt{T} = 1.3415$; $\mathrm{N}(d_2) = 0.9101$.
$E_0^{''} + C^* = (240 + 26.68)\mathrm{N}(d_1) - 226.69e^{-0.05(1)}\mathrm{N}(d_2)$ = 248.57 – 196.25 = 52.32.

Figure 4 – Plausible scenarios and balance sheets for the financial institution



Note: Figure 4 shows how the FI's balance sheet evolves under "healthy", "strained", and "recapitalized" scenarios. C^* stands for capital injection for acquiring 51% ownership in the recapitalized FI. The detailed calculations for each scenario are contained in subsection 3.1, 3.2, and 3.3.

4. DISCUSSION AND IMPLICATIONS

Our discussion relates to scenarios appearing in Figure 4 and to the wide range of asset value situations in Table 1. Its emphasis is on wealth and default effects of changes in asset values, the limits of recapitalizations by private investors, and the implications of government bailouts when private capital becomes unavailable.

4.1 Wealth Effects

Reconsider first the wealth effects associated with the recapitalization in the 3rd scenario of Figure 4 wherein private capital $C^* = 26.68$ was injected. As Figure 4 shows, the injection causes debt to become less risky. Accordingly, debt value gains 3.48, from $D_0' = 210.88$ to $D_0'' = 214.36$. As for the entity's asset value, which had first dropped 20% from $V_0 = 300$ to $V_0' = 240$, it rebounded by C^* to $V_0'' = V_0' + C^* = 266.68$, whereas boosted equity equalled $E_0'' + C^* = 25.64 + 26.68 = \frac{C^*}{0.51} = 52.32$. In parallel, the initial equity value $E_0 = 84.55$ suffered first a 66% meltdown to $E_0' = 29.12$, due to the 20% drop in asset value. It further declined by 3.48 to $E_0'' = 25.64$ due to debt gaining in security and value as a result of injection C^*. In toto, the existing FI shareholders lost

58.91, or 70% of their stake value, besides losing capital ownership control, which passed to the outside capital provider at the 51% level.

The natural question to ask here is why the existing FI shareholders would ever accept such a bailout? This has to do with restoring public confidence in the FI. At a time of great uncertainty, given that important FIs often possess an array of complex, and thus opaque, structured products (such as CDOs), a downward spiral of asset values can develop and erode further the wealth of existing shareholders.[10] Understandably, they will seek recapitalization so as to improve the FI's prospects of survival and prevent further decline in asset value. Moreover, the bailout initiative in itself, because it is costly to existing shareholders, sends a clearly positive signal to the market about the seriousness and resolve of the FI's management.

4.2 Effect on Default Probability

Notable is that injection $C^* = 26.68$ has reduced default probability p, from a worrisome 20.08% to a livable 6.16%. Had the outside investor aimed at a safer p of 1%, his injection would have been $C = 60$ so as to bring asset value from 240 to 300, the level which commands 1% p, as shown in first row of Table 1.[11] In our context, injecting capital C means adding C to asset value and the latter becomes the key to calculating p. Similarly, withdrawing W in capital would reduce asset value by W and increase p accordingly. *In essence, assuming constant asset volatility, the principle behind adjusting default probability p to its target level is to choose C, or W, so as to bring about an asset value* V_0 commensurate *with the p target*. In Table 1, for instance, the C additions necessary to get to the health p of 1%, starting from any asset value V_0, would equal $C^{\mathrm{H}} = 300 - V_0$. So in rows 1, 2, 3, etc., the $C^{\mathrm{H}}s$ would be 0, 15, 33.32, etc. Had shareholders in row 2 situation with $V_0 = 285$ and $p = 2.36\%$, been distributed a large dividend W totalling 30, the FI's asset value would have dropped to 255, and p to 10.70%. Of course, withdrawals large enough to increase default probability p sizeably are rather uncommon. Entities with highly depressed asset values would rarely attract new capital.

Table 1 – Behaviors of equity (*E*$_0$), debt (*D*$_0$), default probability (*p*), and capital injection (*C*) with changes in asset value (*V*$_0$) of a financial institution

V_0	$E_0 = K_p$	D_0	$p = \mathrm{N}(-\hat{d}_2)$	C^{H}	C^{+}	Feasibility
300.00	84.55	215.45	1%	0	n.a.	n.a.

[10] The collapsing asset prices of many U.S. financial institutions at the beginning of the bank crisis in 2007 are vivid examples of such downward spiral of asset values.

[11] Note in passing that with outside $C = 60$, V_0 restored to 300, and p to 1%, the resulting equity would equal $E_0 = 84.55$, leaving 84.55 – 60.00 = 24.55 to existing shareholders, compared to 25.64 (= 52.32 – 26.68) when $C = 26.68$. Their loss of 25.64 – 24.55 = 1.09 equals the gain in debt value due to p improving from 6.16% to 1%.

285.00	69.82	215.18	2.36%	15.00	n.a.	yes
266.68	52.32	214.36	6.16%	33.32	n.a.	yes
255.00	41.69	213.31	10.70%	45.00	n.a.	yes
240.00	29.12	210.88	20.08%	60.00	n.a.	yes
236.20	26.20	210.00	23.20%	63.80	n.a.	yes
225.00	18.38	206.62	34.15%	75.00	n.a.	yes
215.45	12.79	202.66	45.25%	84.55	84.55	threshold
210.00	10.10	199.90	52.05%	90.00	26.20	no
195.00	4.61	190.39	70.73%	105.00	6.73	no
180.00	1.64	178.36	85.97%	120.00	1.90	no

Note: The table shows how risk capital (K_p) or equity cushion (E_0), debt value (D_0), and default probability (*p*), vary with asset value (V_0). $D_\mathrm{T} = 226.69$ is the debt due at time T = 1. Expected growth rate of asset value $\mu = 8\%$, asset value volatility $\sigma = 15\%$, and risk-free interest rate $r = 5\%$.

$K_p = V_0\mathrm{N}(d_1) - D_\mathrm{T}e^{-r\mathrm{T}}\mathrm{N}(d_2)$, where: $d_1 = \frac{\ln(V_0/D_\mathrm{T})+(r+0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$, and $d_2 = d_1 - \sigma\sqrt{\mathrm{T}}$; where: N(*d*) is probability in the N(0,1) *z*-table that *z* < *d*.

p is default probability measured by N($-\hat{d}_2$), where:. $\hat{d}_2 = \frac{[\ln V_0 + (\mu-0.5\sigma^2)\mathrm{T}]-\ln D_\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$.

$C^\mathrm{H} = 300 - V_0$ is the capital needed to restore *p* to the healthy level of 1%.

C^+ is the limit of subsidy-free capital injection. If $C^\mathrm{H} < C^+$, then subsidy-free capital injection is feasible; otherwise, it is infeasible.

4.3 The Limit to Private Capital Injection (C^+)

Recall that the private equity investor wants his full money's worth of the shares acquired. This entails that the buffer of existing equity is thick enough to cover the subsidy-like wealth transfer to debtholders. Thus, for a given pre-injection asset value $V_0 = D_0 + E_0$, there exists a critical/maximum *C*, say C^+, that will cause a wealth transfer to debtholders that exactly wipes out the existing shareholders' wealth E_0. This scenario is illustrated in Table 1 with initial $V_0 = 210, D_0 = 199.90$, and $E_0 = 10.10$. A capital injection $C^+ = 26.20$ will result in asset value $V_0^{'} = V_0 + C^+ = 210 + 26.20 = 236.20$, equity value $E_0^{'} = 26.20 = C^+$, and debt value increase $\varDelta D_0 = D_0^{'} - D_0 = 210 - 199.90 = 10.10$ which exactly wipes out the initial equity E_0. Had the injection equaled C = 30 instead of 26.20, the outcome would have been $V_0^{'} = 210 + 30 = 240$, $E_0^{'} = 29.12$, $\varDelta D_0 = 210.88 - 199.90 = 10.98$, a two-pronged wealth transfer would have occurred: 10.10 which depletes initial shareholders' equity value, and the remainder, 0.88, which erodes the value of the capital provider's stake in the resulting equity. So he ends up with an equity value of 29.12, which is inferior to his capital injection of 30.00, a situation he would find unacceptable of course. So assuming the investor could anticipate such an outcome, he would have limited his capital injection to $C^{+} = 26.20$.

Thus, given that the maximum acceptable capital injection C^{+} represents the whole FI's post-injection equity, then for each of its asset value V_0, the limit C^{+} is the solution to the following option-based equation for equity value:

$$C^{+} = (V_0 + C^{+})\,\mathrm{N}(d_1) - D_{\mathrm{T}}e^{-r\mathrm{T}}\mathrm{N}(d_2) \qquad \text{---- (22)}$$

where $d_1 = \frac{\ln[(V_0+C^{+})/D_{\mathrm{T}}]+(r+0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$, and $d_2 = d_1 - \sigma\sqrt{\mathrm{T}}$.

Any capital injection larger than C^{+} would entail both erasing the existing shareholders' equity and subsidizing debtholders.

As for finding the maximum acceptable C^{+}, an alternative approach could be to determine the minimum acceptable asset value V_0 for a given size of capital injection C^{+}. This minimum asset value, V_0, is the solution to the same option-based equation: $C^{+} = (V_0 + C^{+})\,\mathrm{N}(d_1) - D_{\mathrm{T}}e^{-r\mathrm{T}}\mathrm{N}(d_2)$. For example, if a recapitalization plan comes with a fixed amount of equity capital $C^{+} = 30$, then the minimum asset value V_0 for securing such a capital injection is 211.11, as solved from the above equation. Thus, any FI with an asset value below 211.11 will not be aided with a 30 equity capital infusion, unless the capital provider is ready to offer a subsidy to the FI and/or debtholders are willing to concede some value with a debt restructuring.

The three rightmost columns of Table 1 provide more results on feasibility of a subsidy-free capital injection for each asset value V_0. $C^{\mathrm{H}} = 300 - V_0$ is the capital injection needed to restore an FI's insolvency probability to a targeted healthy level of 1%. C^{+} is the capital injection that exactly wipes out existing shareholders' equity value. If $C^{\mathrm{H}} < C^{+}$, then it is feasible for a subsidy-free capital injection to restore the FI's p to the healthy level of 1%; otherwise, it is infeasible. As the table shows, when asset value falls below a threshold level of 215.45, no deal is warranted.

4.4 Transposing to Public Bailouts

It is common occurrence that a rather distressed FI seeks a government bailout. Generally, it has seen its asset value plummet beyond a level that precludes private recapitalization. It still aspires to being bailed out rather than restructured or liquidated. Our option approach enables us to perceive its situation.

Consider the FIs in Table 1 (bottom rows), whose anterior asset value of 300 has fallen to distress V_0 levels of say 195 or 180. The wealth of its initial stockholders would have plunged from 84.55 to as low as 4.61 or 1.64, respectively. The FI would likely search in vain for private capital because a good part of this capital would be eroded in favor of debtholders. As more equity pours in, debt absorbs more value. For example, with $V_0 = 195$ and default p around 71%, a capital injection of 60 would mean new $V_0 = 195 + 60 = 255$ and $p = 10.70\%$. But, as can be derived from Table 1 rows with 195 and 255 as asset values, debtholders would gain $\Delta D = 213.31 - 190.39 = 22.92$ in the process. Thus the injection would not only wipe out the initial equity value of 4.61, but also erode the private investor's stake by 18.31 (= 22.92 – 4.61), such that the capital input of 60 would result in a new equity value of E_0= 41.69. Obviously, the cautious private investor would not be willing to inject 60, because, in theory at least, he stands to lose wealth.

In principle, the same caution should be exercised in public financing or bailouts. It happens though that government cannot ignore, unlike the private sector, strong public pressure when it comes to either rescue hardly viable entities, including FIs, in periods of crisis, or attract new firms by indirect subsidies, in easy-loan form for instance, or pure subsidies. In sum, it is another ball game. We still can ask who are the first-hand beneficiaries.[12] For sure, debtholders profit since their stake becomes less risky and more valuable. Existing shareholders can gain also provided they avoid squandering the "free lunch" served them. We can understand the advice of laissez-faire economists against government subsidies. There is much coherence in the rules they uphold. Possibly, too much for an unruly world. Hence debates that never end as time goes by.

4.5 Bailout versus Bankruptcy

As already shown by quantified examples, a highly indebted FI becomes desperate if its asset value falls to low levels. Then no private investor will want to inject capital blindly since it would serve in good part to subsidize debtholders. The normal outcome is bankruptcy with reorganization, or partial compensation for debtholders through sales of assets. In the U.S., the Federal Deposit Insurance Corporation (FDIC) has the power to liquidate FDIC-insured failing FIs in ways respecting conventional economic theory. However, its action can be seen at best as supplement to government cosly bailouts when a massive wave of failing FIS is about to materialize, as was the case in the wake of the subprime lending crisis. Nonetheless, the question of keeping alive desperate firms through government subsidies at the expense of taxpayers remains controversial. Why should existing shareholders and even junior debtholders, those investor categories that stand to gain more from taking more risk, be aided when losses are their lot?

On the one hand, able independent financial economists, through widely-accessible forums, such as Fama and French (2009), often stigmatize the government when it steps in to save FIs and firms in general. They preach conventional economic beliefs already alluded to. On the other hand, just as able government economists justify using tax-money for bailouts. Invoking political pressure for such backing is a debatable argument. A valid question is: under what redeeming conditions government bailouts might be appropriate? In theory, governments should avoid subsidizing firms and let them fail. In practice, especially when giant domino effects threaten to create havoc in critical sectors, such as banking and insurance, governments generally assent to intervene in order to contain the ominous crisis and its contagion.

[12] In the case of commercial banks, the bulk of debts comes in the form of savings of depositors. If the priority of a government bailout is to safeguard the savings of millions of small depositors, then subsidies to the FIs may be justified.

Witness the Troubled Asset Relief Program (TARP), this major bailout initiative initiated by the U.S. Treasury and Federal Reserve in late 2008 and terminated officially in late 2010. The intent was to stabilize still healthy, rather than failing, FIs. However, it soon faced a real challenge in merely distinguishing between the two types of FIs, both effectively (without type errors) and efficiently (within a short time period). The various scenarios studied above exemplify how recapitalization is meant to improve the health of FIs rather than favor increased lendings as commonly suggested in many political debates.[13] This primary financial health concern comes out in the capital injection case of Figure 4, (C = 26.68) with a view to obtain 51% control. It shows how total equity value can be sizably increased from $E_0^{'} = 29.12$ to $C + E_0^{''} = 26.68 + 25.64 = 52.32$ to support the existing debt obligation of 226.69, which reduces default probability p from roughly 20% to a healthier 6%. It shows also that private capital deals will be closed provided the value of the equity stake acquired equals the amount of capital invested.

Our calculations and discussions above indicate that government-sponsored bailout plans, such as TARP, can potentially improve the financial health of fragilized FIs. There is no denying that timely capital injections can only but contribute to lower default probability (p) for institutions, and thus improve their solvency. A low p must remain a constant target for any FI. It also fuels public confidence in the FI. Obviously, should tax money be justified for bailout, it should be channeled towards troubled, yet viable, FIs. In theory, government should refrain from bailing out FIs. In practice, bailouts make real sense during severe long-lasting crises when the private sector is threatened by contagious financial disasters.

5. CONCLUSION

Our paper touches on the persistent, crisis-related, issue of recapitalizing troubled financial institutions (FIs). We adopt an approach permitting to view equity capital as an option on the FI's underlying asset value distribution. We endeavor to present the basic premises and technical concepts involved in an intuitively appealing way. The intent is to facilitate their pedagogical use in finance courses. Numerous quantified examples are provided with a double aim. First, we see fit to clearly show that the approach can be applied to capital-structure decisions, especially in FIs whose primary solvency-related concern is to keep default probability low. Second, it is worth stressing that the approach reveals the interrelated effects of capital injections on both debt and equity values.

Beyond showing how capital injections by private investors or governments in strained FIs can restore their financial health, the no-subsidy principle is recalled. Capital providers should get their money's worth in new securities and let initial shareholders, and even junior debt holders, bear their losses.

The practical relevance of the option approach cannot be timelier than during a financial crisis when solvency concerns become pervasive. This is the case since late

[13] We can find on the web abundant discussions or forums on the pros and cons of government-sponsored rescue plans adopted in the wake of Lehman Brothers' bankruptcy. See Zaghini (2009).

2007 when an engulfing financial crisis rocked most economies of the world, a crisis that has yet to recede convincingly.

Obviously, the option approach to solvency-related issues in FIs is more theoretically satisfying than the static theory of trade-off between tax benefits and bankruptcy costs. Its theoretical advantage lies in the plausible interrelated values it can provide for FIs, or firms in general. More explicitly, the option approach permits to intuit how the dynamic behavior of asset values impacts on solvency via its interrelated effects on debt and equity. We believe the approach can only but help the pedagogue bent on leaving students with a solid intuition, if not grasp, about solvency issues. Further, the option approach is appealing since it connects directly with the basic principle of finance, that is: *Securities, including options, are worth the discounted value of their expected net cash flows*. This is so because securities amount fundamentally to conditional claims on underlying future value distributions.

Of course, the option approach to solvency issues gains perhaps in realism if it relies on complex capital structure models of the kind mentioned in our introduction. However, complexity may harm rather than help students develop, as aimed at, a rich intuition about the issues involved. Further, it so happens that the constant asset volatility hypothesis, on which our exposition hinges a lot, finds empirical support for the class of highly leveraged firms. And FIs, by nature, top this class in terms of leverage.[14] Yet, because it makes sense perceptually to expect asset volatility increases in FIs subjected to sharp drops in asset value, we examined how our results change by relaxing the assumption of constant asset volatility. We conducted sensitivity analyses under various hypotheses associating higher asset volatilities with larger drops in asset value.[15] Obviously, ascribing increased asset volatility to lower asset values yields different levels for risk capital, debt value, and default risk. But the lessons remain the same, namely, 1) default probability improves upon capital injection; 2) the existing shareholders bear the brunt of falling asset values and stand to lose further with an external capital injection due to wealth absorption by debt securities; and 3) there is a limit to private capital injection which is reached when the injection equals the wealth of existing shareholders.

Other extensions may include combining various debt horizons into an horizon equal to duration, relaxing the fixed expected asset value growth, and studying the relationship between asset, debt, and equity volatilities. Exploring these issues, however, will go beyond the scope of our paper which is intended to be pedagogical in nature.

[14] See Choi and Richardson (2008, Figure 1, p. 29 in particular) and Choi and Richardson (2012, Figure 1, p. 2 in particular). Care must be taken to translate typical monthly volatilities into annual volatilities in continuous time. Thus, the stable periodic volatility of 6% per month becomes approximately a volatility of $\ln(1 + 0.06\sqrt{12}) = 18.88\%$ per year in continuous time – a magnitude very close to the 15% asset volatility we used throughout our paper.

[15] The sensitivity results are available from the authors upon request.

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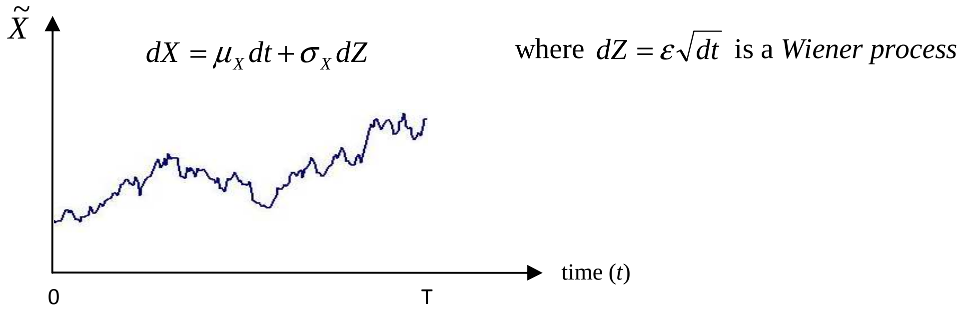
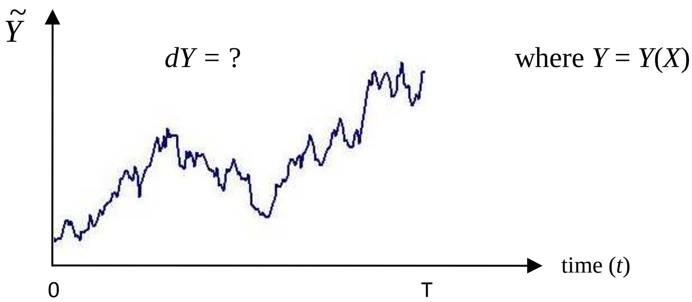
APPENDIX
Derivation and Illustration of GBM, Ito's Lemma, Lognormal distribution, and *p*th quantile

A stochastic process is a dynamic process describing how a variable evolves over time. Itô's Lemma tells us how to obtain the mathematical expression of a variable Y that is a function of a variable following another stochastic process, as illustrated in the following graphs:





From Itô's Lemma,

$$dY = \left(Y_t + Y_X \mu_X + \frac{1}{2} Y_{XX} \sigma_X^2 \right) dt + Y_X \sigma_X dZ$$

or

$$dY = \left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial X} \mu_X + \frac{1}{2} \frac{\partial^2 Y}{\partial X^2} \sigma_X^2 \right) dt + \frac{\partial Y}{\partial X} \sigma_X dZ . \qquad \text{---- (A1)}$$

If $\mu_X = 0$ and $\sigma_X = 1$, then $dX = dZ$, so the stochastic process of variable *X* is a Wiener process. In this case, *dY* is a direct function of the Wiener process *dZ*, and Ito's Lemma, equation (a), becomes

$$dY = \left(\frac{\partial Y}{\partial t} + \frac{1}{2} \frac{\partial^2 Y}{\partial Z^2} \right) dt + \frac{\partial Y}{\partial Z} dZ . \qquad \text{---- (A2)}$$

Thus, if a variable *Y* (any variable *Y*, including asset value *V* -- our variable of interest) is a function of Wiener process *dZ*, then the stochastic process of *Y* is given by equation (A2).

Now, we turn to the variable -- asset value, *V*. In equation (1), we have modeled the *V* as following a geometric Brownian motion (GBM) process, which is a special stochastic process, thus, Itô's Lemma, equation (A2) applies to *V*. By comparing equation (1) and equation (A2), we can conclude that the variable *V* must satisfy the following two conditions:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial Z^2}=\mu V \quad \text{and} \qquad \text{---- (A3)}$$

$$\frac{\partial V}{\partial Z}=\sigma V\ . \qquad \text{---- (A4)}$$

We write equation (A4) as

$$\frac{\partial V}{V}=\sigma\partial Z$$

By take integration for the above expression, we get

$$\ln V=\sigma Z+f(t)$$

or $$V=e^{\sigma Z+f(t)}\,. \qquad \text{---- (A5)}$$

Differentiating the above expression with respect to *t* and *Z*, respectively, we obtain

$$\frac{\partial V}{\partial t}=V\frac{df}{dt}, \qquad \frac{\partial V}{\partial Z}=V\sigma, \qquad \text{and} \qquad \frac{\partial^2 V}{\partial Z^2}=V\sigma^2.$$

We insert these identities into equation (A3) to get

$$V\frac{df}{dt}+\frac{1}{2}V\sigma^2=\mu V\ ,$$

i.e., $$\frac{df}{dt}=\mu-\frac{1}{2}\sigma^2\,.$$

Integrating the above expression from 0 to t, we obtain

$$f=\int_0^{\mathrm{t}}\left(\mu-\frac{1}{2}\sigma^2\right)dt=\left(\mu-\frac{1}{2}\sigma^2\right)\mathrm{t}+\mathrm{c}\,,$$

where *c* is a constant of integration.

Substituting the above expression of *f* into equation (A5),

$$V_{\mathrm{t}}=e^{\sigma Z+(\mu-0.5\sigma^2)\mathrm{t}+c}=e^{c}\times e^{(\mu-0.5\sigma^2)\mathrm{t}+\sigma Z} \qquad \text{---- (A6)}$$

To choose the constant *c*, suppose that at t = 0, *Z* = 0. So, $V_0=e^c$.

$$V_{\mathrm{t}}=V_0 e^{(\mu-0.5\sigma^2)\mathrm{t}+\sigma Z}$$

or $$\ln V_{\mathrm{T}}=\ln V_0+(\mu-0.5\sigma^2)\mathrm{T}+\sigma Z_{\mathrm{T}} \qquad \text{---- (A7)}$$

Note that the variable Z_{T} in equation (A7) is from a Wiener process and distributed normally at time T with expected value of 0 and variance of T. In fact, equation (A7) is just equation (8) with $Z_{\mathrm{T}} = \sqrt{\mathrm{T}}\varepsilon$, so the variable $\ln V_{\mathrm{T}}$ in equation (A7) is a normal variable with the expected value

$$E(\ln V_{\mathrm{T}}) = \ln V_0 + (\mu - 0.5\sigma^2)\mathrm{T} \qquad \text{---- (A8)}$$

and variance

$$Var(\ln V_{\mathrm{T}}) = \sigma^2 \mathrm{T}\,. \qquad \text{---- (A9)}$$

Results in equations (A7) - (A9) are precisely what equations (3) - (5) contain.

Thus, we have proven that if *V* follows a geometric Brownian motion process, then *V* at a terminal time T is distributed lognormally.

Finally, we turn to the p^{th} quantile of a lognormal distribution. While we have purposely equated the p^{th} quantile of asset value V_{T} to debt liability D_{T} in equation (7) to get the default probability, the general expression for p^{th} quantile is readily available from equation (6) by replacing D_{T} with p^{th} quantile. For an arbitrary probability *p*,

$$p\ (V_{\mathrm{T}} < p^{\mathrm{th}}\text{ quantile}) = \mathrm{N}\left(-\frac{\ln\left(V_0 / p^{\mathrm{th}}\text{quantile}\right) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}\right). \qquad \text{---- (A10)}$$

If we denote N^{-1} as the *inverse* of the cumulative unit normal distribution function, N, then by definition, N^{-1}[N(x)] = x. For example, the most commonly seen N^{-1}(1%) = - 2.33 and N^{-1}(5%) = - 1.65. By applying N^{-1} to both sides of equation (A10), we obtain

$$\mathrm{N}^{-1}(p) = -\frac{\ln\left(V_0 / p^{\mathrm{th}}\text{quantile}\right) + (\mu - 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$$

i.e. $\mathrm{N}^{-1}(p)\,\sigma\sqrt{\mathrm{T}} = -\ln\left(V_0 / p^{\mathrm{th}}\text{quantile}\right) + (\mu - 0.5\sigma^2)\mathrm{T}$

Now we take exponentiation on both sides of the above expression and rearrange the terms:

$$p^{\mathrm{th}}\text{ quantile} = V_0 e^{(\mu - 0.5\sigma^2)\mathrm{T} + \sigma\sqrt{\mathrm{T}}\mathrm{N}^{-1}(p)}\,. \qquad \text{---- (A11)}$$

Equation (A11) is the general expression for the p^{th} quantile of a lognormal distribution.

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Excel Spreadsheets Illustrating Calculations

Capital Infusion (C) for majority stake (51%) Calculation using Solver

Outputs:

C	$26.68

Solver: equation (21) (0.000000) <= (use Solver at cell G7 here:
click cell G7;
go to Tool to get Solver;
click Solve.
then value for C is solved)
What shown in cell G7 is the specified
Precision in the Options of Solver

Inputs:

σ	15%
D_T	226.69
r_f	5%
T	1

Calculations:

$d1$	1.491476
$d2$	1.341476
N($d1$)	0.932082
N($d2$)	0.910117

$$\frac{C}{0.51} = (240 + C)\mathrm{N}(d_1) - D_T e^{-r_f T}\mathrm{N}(d_2) \quad \text{equation (21)}$$

$$d_1 = \frac{\ln\left((240 + C)/D_T\right) + (r_f + 0.5\sigma^2)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$$

$$d_2 = d_1 - \sigma\sqrt{\mathrm{T}}$$