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CO-INTEGRATED COMMODITY FORWARD PRICING MODEL

Abstract:

Commodities pricing needs a specific approach as they are often linked to each other and so are expectedly doing their prices. They are called co-integrated when at least one stationary linear combination exists between them. Though widespread in economic literature, and even if many equilibrium relations and co-movements exist in economy, this principle of co-movement is not developed in derivatives field. Present study focuses on the following problem: How can the price of a forward agreement on a commodity be simulated, when it is co-integrated with other ones? Theoretical analysis is developed from Gibson-Schwartz model and analytical solution is given for short maturities contracts and under risk-neutral conditions. Application has been made to crude oil and heating oil energy commodities and result confirms the applicability of proposed method.

Keywords:

Co-integration, Commodities, Forward Pricing, Gibson-Schwartz.

JEL Classification: C32, D40

I-INTRODUCTION

Economic studies have been for long concentrated on existence of interrelations between various objects and entities belonging to this field in order to build up a reliable model of economic ensemble dynamics such as a market. Unlike Mechanics however, many variables in such system are difficult if not possible to even identify, and more modest approaches trying to relate the trends of specific, generally global, variables have been developed [1]. These macro-models are limited and need to be complemented, like in classical Physics case, by more detailed micro-analysis models often providing realistic evaluation of their parameters. Another interesting approach has also been investigated concerning the determination, from collected data observations, of existing links and their strengths between system elements. This “inverse” problem is quite useful for reconstitution of interaction network representing any system dynamics. In present case, attention is focused on dynamical similarity of market elements and, more precisely here, on commodities. It is intuitively understandable that such instruments should behave in non-independent way, as finance holders can easily switch from one to another one to respond to market move and even amplify it. Basically, the conceptual issue can be represented by a simple mechanical analogy. Given two weighting balls A and B linked up by a spring K, under market action, each ball follows a random path with dispersion radius R_A, R_B see Figure 1. Each ball represents here a commodity and K symbolizes the correlation force between them each having volatility σ_A, σ_B equivalent to radius R_A, R_B.

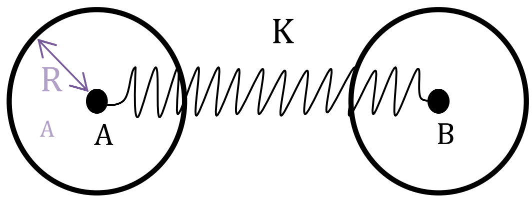

Figure 1. Mechanical Analogy of Commodities Dynamics

There is competition between global converging K force and “microscopic” local fluctuations acting supposedly in a random way on A and B. Depending on looseness (or stiffness) of the two forces, dynamical behaviors of A and B will be loosely (or stiffly) related. A direct model of system dynamics can be set up in order to predict possible linkage between A and B motions, and analytically solved in linear case [2]. But another point is conversely to reconstitute spring force K from observation of dynamical evolution of A and B, in order to document the interaction matrix regulating exchanges between the various market elements.

In economic domain, analysis will be similarly developed for very nearby commodities for which equivalent spring force K is very stiff. There exists a strong correlation between their dynamics usually modeled by co-integration principle [3] with evident consequence on prices. As energy markets have been considerably growing recently, valuation of energy-linked financial instruments represents an important issue for research and market trade [4]. Because a commodity can be consumed, its price is a combination of future asset and current consumption values. So unlike financial derivatives, storage of energy products is costly, and sometimes practically impossible like for electricity or carbon rights. Consequently, physical ownership of a commodity carries an associated flow of services. With flexibility option on consumption (no risk of commodity shortage), the decision to postpone it implies storage expenses. So commodity price is due to the impact of combined production, inventory levels and storage. These factors imply a deviation from normally held price, represented by a “convenience yield” term [5] symbolized by the effect of spring force K. The problem considered here is the simulation of commodity forward contract when this commodity is in commodity futures, and options are priced in a two-factor model with spot commodity prices and mean reverting convenience yields [6]. Corresponding GSC model belongs to the broad class of "spot" models [7]. A spread of GS work is proposed in [8] where GSC model is extended by introducing linear relations that commodity prices should satisfy, including co-integration under certain conditions. The co-integration model is analytically developed in next paragraph, and because data are already collected, application in energy commodity sector will be devoted to the fairly connected pair of heating oil and crude oil [9]. Extension to the case of looser spring linkage between commodities, for instance between two metals such as copper and iron which are close enough but farther apart than previous pair, will be considered elsewhere.

II-THE MODEL

The principle of co-integration allows identify relation between different variables. This statistical property concerns time series variables. If two or more series are individually integrated and some linear combination of these series has a lower order of integration, then the series are said to be co-integrated. Here it is assumed that spot price vector $\mathbf{S}(t)$ and convenience yield vector $\boldsymbol{\delta}(t)$ of commodities i (i=1,2,..n) obey the differential system [10]

$$d\,\mathbf{X}(t) = [\mathbf{r}-\boldsymbol{\delta}(t) + \mathbf{b}z(t)]dt + \boldsymbol{\sigma}_S.d\mathbf{W}_S(t) \quad (1)$$
$$d\boldsymbol{\delta}(t) = \boldsymbol{\kappa}[\boldsymbol{\alpha}-\boldsymbol{\delta}(t)]dt + \boldsymbol{\sigma}_\delta.d\mathbf{W}_S(t)$$

with $\mathbf{S}(t) = [S_1(t),S_2(t),..S_n(t)]^T$, $\mathbf{X}(t) = \mathrm{Ln}\,\mathbf{S}(t)$, $\boldsymbol{\delta}(t) = [\delta_1(t), \delta_2(t), .. \delta_n(t)]^T$, $\mathbf{r} = [r,r,\ldots r]^T$ where r is risk-free interest rate assumed to be constant. Adjustment speeds $\mathbf{b} = [b_1, b_2,.. b_n]^T$, convenience yield parameters $\boldsymbol{\kappa} = \mathrm{diag}[\kappa_1,\kappa_2,..\kappa_n]$, $\boldsymbol{\alpha} = [\alpha_1,\alpha_2,..\alpha_n]^T$, convenience yield volatility $\boldsymbol{\sigma}_\delta = \mathrm{diag}[\sigma_{\delta 1}, \sigma_{\delta 2},.. \sigma_{\delta v}]$, and spot price volatility $\boldsymbol{\sigma}_S = \mathrm{diag}[\sigma_{S1},\sigma_{S2},..\sigma_{Sn}]$ are constant coefficients vectors and matrices. $\mathbf{W}(t) = [\mathbf{W}_S(t),\mathbf{W}_\delta(t)]^T = [WS1(t)...WSn(t), W\delta 1(t)...W\delta n(t)]^T$ is the 2n dimensional Brownian motion under the risk-neutral probability with correlations

$$dWSi(t)dWSj(t') = \rho_{SiS}j\ \delta(t-t')dt,\ dWSi(t)dW_{\delta j}(t') = \rho_{Si\delta j}\ \delta(t-t')\,dt,\ dW_{\delta\iota}(t)dW\delta j(t') = \rho_{\delta\iota\delta j}\ \delta(t-t')dt \quad (2)$$

For commodities, the global nature of their market usually leads them to be linked to each other, and so do their prices. Hence, at least one stationary linear combination of them exists (even if they are individually non-stationary), that is to say they are co-integrated. So it is assumed that there exists between commodity prices a linear stationary combination

$$z(t) = \mu_z + a_0 t + \mathbf{a.X}(t) \quad (3)$$

where $\mu_{z,}$ a_0, and $\mathbf{a}= [a_1, a_2,.. a_n]^T$, are unobservable constants [11]. Integration of equations (1) gives

$$\mathbf{X}(T) = [\mathbf{r} -.5\boldsymbol{\sigma}_S{}^2](T-t) + \mathbf{X}(t) + \mathbf{b}Z(T,t) - \Delta(T,t) + \boldsymbol{\sigma}_S.\int_t^T d\mathbf{W}_S(u)du \quad (4)$$
$$\boldsymbol{\delta}(u) = \mathbf{E}(u-t)\ \boldsymbol{\delta}(t) + [\mathbf{I}- \mathbf{E}(u-t)]\ \alpha + \sigma_\delta.\int_t^u \mathbf{E}\ (s-t)d\mathbf{W}_\delta\ (s)$$

in terms of time integrals Z(T,t) and Δ(T,t) of z(t) and δ(t) respectively which contain random terms from dW(.,.) terms and where **E**(.) = diag[E_1(.),E_2(.),..E_n(.)], $E_j(s) = \exp[-\kappa_j s]$. After manipulation it is possible to write **X**(T) in the form

$$\mathbf{X}(T) = \mathbf{X}(t) + \mathbf{D}(T,t) + \boldsymbol{\sigma}_S . \int_t^T d\mathbf{W}_S(u)du - \boldsymbol{\sigma}_\delta . \int_t^T \psi(T-u,\boldsymbol{\kappa},0)d\mathbf{W}_{\boldsymbol{\delta}}(u) \quad (5)$$

where explicit stochastic terms have been singled out and deterministic terms are collected in $\mathbf{D}(.,.) = [D_1(.,.), D_2(.,.),.. D_n(.,.)]^T$ and with $\psi(x,\boldsymbol{\kappa},B) = (B\mathbf{I}+\boldsymbol{\kappa})^{-1}[\mathbf{E}(x) - E_0(x)\mathbf{I}]$, $E_0(x) = \exp(Bx)$, B = **a.b**. One then gets

$$Z(T,t) = m(T-t) + [z(T) - z(t)]/B + (\mathbf{a}/B).\ \Delta(T,t) - (\mathbf{a}/B)\ \boldsymbol{\sigma}_S . \int_t^T d\mathbf{W}_S(u)du \quad (6)$$

$$\Delta(T,t) = [(T-t)\mathbf{I} - \psi(T-t,\boldsymbol{\kappa},0)].\boldsymbol{\alpha} + \psi(T-t,\boldsymbol{\kappa},0).\delta(t) + \boldsymbol{\sigma}_\delta . \int_t^T \psi(T-u,\boldsymbol{\kappa},0).\, d\mathbf{W}_{\boldsymbol{\delta}}(u)$$

with $m = -(1/B)[a_0 + \mathbf{a.r} - .5\ \mathbf{a.}\boldsymbol{\sigma}_S^2]$ and $\boldsymbol{\sigma}_S^2 = \mathrm{col}[\sigma_{S1}^2,\sigma_{S2}^2,..\sigma_{Sn}^2]$.

III-FORWARD CONTRACT EVALUATION

From previous expressions, it is possible to express forward contract price with maturity T and horizon t given by **F**(T,t) = < **S**(T,t)> and where < > represents random averaging in usual form

$$\mathbf{F}(T,t) = \exp\langle\mathbf{X}(T,t)\rangle.\exp.5\langle[\mathbf{X}(T,t) - \langle\mathbf{X}(T,t)\rangle]^2\rangle \quad (7)$$

which requires evaluation of expectation and variance of random function **X**(T,t). As it is observed from (4,5,6) **X**(T,t) includes a deterministic term and a stochastic one implying full random vector **W**(t) so calculation of $\mathrm{Var}\{\mathbf{X}(T,t)\} = \langle[\mathbf{X}(T,t) - \langle\mathbf{X}(T,t)\rangle]^2\rangle$ utilizes all cross correlations (2). Difficulty is to find a set of independent base functions on which fluctuating part **X**(T,t) – <**X**(T,t)> can be projected. Here analysis of the various stochastic integrals in (5) shows that for each commodity i there are five distinct terms which can be reorganized as

$$Y_i^m(T,t) = \int_t^T K_i^m(u)dW_{m,i}(u) \quad (8)$$

where m = 1,..,5 and weighting coefficients $K_i^m(u)$ are

$$K_i^1(u) = K_i^4(u) = 1;\ K_i^2(u) = E_i(-u) - 1;\ K_i^3(u) = K_i^5(u) = E_0(-u) - 1$$
$$dW_{1,i}(\cdot) = dW_{2,i}(\cdot) = dW_{3,i}(\cdot) = dW_{\delta,i}(\cdot);\ dW_{4,i}(\cdot) = dW_{S,i}(\cdot) \quad (9)$$

Representing fluctuating part as $\mathbf{X}(T,t) - \langle\mathbf{X}(T,t)\rangle = \Sigma_{m=1,5}\{\Sigma_{j=1,n}\ C_m^j Y_j^m(T,t)\}$ where all coefficients C_m^j have been explicitly evaluated [12], one finally gets

$$\mathrm{Var}\{\mathbf{X}(T,t)\} = \Sigma_{m=1,5}\{\Sigma_{j=1,n}[C_m^j]^2\ \mathrm{Var}\{Y_i^m\}\} + 2\Sigma_{m=1,5}\Sigma_{j=1,n}\ \Sigma_{p=1,5}\Sigma_{k=1,n}\ C_m^j C_k^p \mathrm{Cov}[Y_j^m Y_k^p] \quad (10)$$

where $\mathrm{Cov}[Y_j^m Y_k^p] = \mathrm{Tab}|\Gamma_{jk}|$ represents the correlation between commodities j and k, ie from (2,8), is given by

$$\Gamma_{jk} = \rho_{mp}\int_t^T K_j^m(u)K_k^p(u)\,du \quad (11)$$

Integrals are elementarily evaluated from definitions (9). Similarly the term <**X**(T,t)> can be explicitly obtained from (5) in the form

$$\langle \mathbf{X}(T,t)\rangle = [\mathbf{r} + m\mathbf{b} - \mathbf{a} + B^{-1}(\mathbf{a}.\alpha)\mathbf{b}](T - t) - \psi(T - t, \boldsymbol{\kappa}, 0)[\alpha - \delta - B^{-1}(\mathbf{a}.(\alpha - \delta))\,\mathbf{b}] + B^{-1}[(m - z)\mathbf{I} + ((B\mathbf{I}+\kappa*)^{-1}\,\mathbf{a}.\,\alpha)]\,\psi(T - t, 0, B)\mathbf{b} - B^{-1}[\mathbf{a}.\delta - ((B\mathbf{I}+\kappa*)^{-1}\,\mathbf{a}.\,\alpha)]\,\psi(T - t, \boldsymbol{\kappa}, B)\mathbf{b} \quad (12)$$

From (10,12) it is possible to calculate forward price of a commodity i knowing actual spot prices and estimated convenience yield parameter of other commodities and to check if the hypothesis that they are in co-integration is justified by observation of market data curves.

III. APPLICATION AND SIMULATION RESULTS

From previous explicit expressions, simulations have been performed for calculating the price of two forward agreements under risk-neutral conditions, based on two supposedly co-integrated crude oil and heating oil commodities [13]. In this case, n=2 and there are 2×5=10 random elements in log spot price representation. Direct covariance matrix can be calculated as in Table 1.

1	0,843748	0,531324	0,000003	0,9636766	0,8131002	0,5120245	2,891E-06	0,5080399	2,879E-06
0,843748	1	0,092128	0,278442	0,8131002	0,9636766	0,0887816	0,26832804	0,08809069	0,26721289
0,531324	0,092128	1	-0,001169	0,5120245	0,0887816	0,9636766	-0,0011265	0,95617722	-0,0011219
0,000003	0,278442	-0,001169	1	2,891E-06	0,26832804	-0,0011265	0,9636766	-0,0011178	0,95967162
0,9636766	0,8131002	0,5120245	2,891E-06	1	0,843748	0,531324	0,000003	0,53112068	2,9997E-06
0,8131002	0,9636766	0,0887816	0,26832804	0,843748	1	0,092128	0,278442	0,09209275	0,27841033
0,5120245	0,0887816	0,9636766	-0,0011265	0,531324	0,092128	1	-0,001169	0,99961733	-0,0011689
2,891E-06	0,26832804	-0,0011265	0,9636766	0,000003	0,278442	-0,001169	1	-0,0011686	0,99988628
0,5080399	0,08809069	0,95617722	-0,0011178	0,53112068	0,09209275	0,99961733	-0,0011686	1	-0,0011689
2,879E-06	0,26721289	-0,0011219	0,95967162	2,9997E-06	0,27841033	-0,0011689	0,99988628	-0,0011689	1

Table 1. Covariance Matrix for Crude Oil and Heating Oil Commodities

The matrix is transformed into positive definite one so that Cholesky decomposition can be applied. It allows obtain the five independent Gaussian components of each vector Y_j (j=1,2) modeling the problem, see Table 2.

-0,75859145	Y_1^1
-0,53097535	Y_1^2
-0,64939027	Y_1^3
0,16831823	Y_1^4
-0,5494394	Y_1^5
-0,24106248	Y_2^1
-0,79571854	Y_2^2
0,11901578	Y_2^3
-0,82431257	Y_2^4
0,11651513	Y_2^5

Table 2. Simulated Vectors Components Y_j^m

As spot price depends on the commodity, each initial spot price has been fixed from the data given in [8], see Figure 2 with WTI and Heating Oil normalized values ($S_1(0)$ =70 ; $S_2(0)$ = 25). It is observed that the two curves are superposed until the end of 2007 when the crisis started, and are proportional after. This is expectable as analysis has been developed here under risk-neutral conditions.

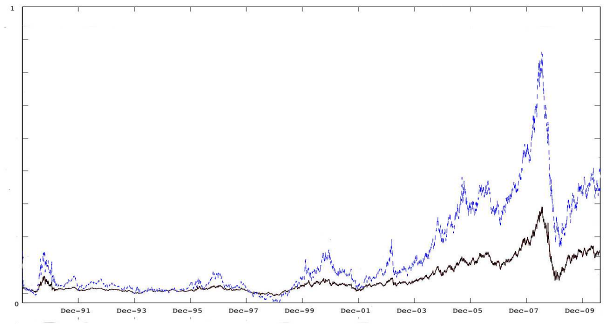

Figure 2**.** WTI and Heating Oil Daily Closing Price from Jan. 2, 1990 to Feb. 27, 2009.
Blue Curve : Price of Heating Oil; Black Curve : WTI Price (multiplied by factor 70/25)

The following figures are obtained see Table 3. Both prices follow the same downward trend. Moreover, they have been multiplied by the same factor here equal to 0.6, highlighting the strong correlation between the two co-integrated commodities [14]

	Spot price **($)**	Forward price G_i (t,T) **($)**
Heating Oil	70	43,3767
WTI	25	15,9142

Table 3. Simulation Results

For completion, the model has been tested for each j $\in \{1,2\}$ with different values of convenience yield δ_i and of volatility σ_i, and it appears that changing these parameters does not have real impact on the final price $G_i(t,T)$.

IV. CONCLUSION

The correlation between “neighboring” commodities in an open market has been represented by using the principle of co-integration. A model using Gibson-Schwartz approach has been proposed to evaluate the strength of relationship between these commodities. This model takes advantage of “convenience yield” which is specific for commodities and represents their storage effect. From mathematical analysis of system equations explicit expression has been obtained for forward price of a contract on a commodity co-integrated with other ones. Application to the (crude oil, heating oil)-couple of commodities for which market data are available has shown their expectable strong inter-relation and relevance of the model for such couple. The result also points to model applicability for short maturities contracts and under risk-neutral conditions which have been taken as working hypothesis, ie typically outside a crisis period for which present analysis should be reconsidered.

AKNOWLEDGMENTS

The authors are very much indebted to ECE for having provided the environment in which the work has been developed and Pr M. Cotsaftis for his help in preparation of the manuscript.

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