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HOW TO MANAGE RISK FOR THE ONLINE ONE-WAY TRADING PROBLEMS?

Abstract:

This paper studies online one-way trading problem, where an investor is given the task of trading dollars to yen. Each day, a new exchange rate is given and the investor must decide how many dollars to convert to yen without knowing the future exchange rates. Since El-Yaniv originally proposed this online problem and presented an optimal threat-based trading strategy, many researchers have been working on innovation based on this model. From the financial risk view, this paper extended El-Yaniv's traditional one-way trading model to present a risk management framework by introducing American put option with the first price as the strike price. This framework extends pure competitive analysis and allows investors to benefit from options. Second, since the option can help the investors to hedge risk, we extend analysis of Al-Binali (1999) to design the option-forecast trading strategy with twice forecasts. The results show that the option-forecast trading strategy constrains the risk of sudden dropping to the minimum price through the American put option. Compared with former research, the competitive ratio of the option-forecast trading strategy is improved effectively.

Keywords:

one-way online trading problem; online algorithm; competitive ratio; option; risk; reward

1 Introduction

In this paper, we focus on the online one-way trading problem, where the investors must make optimal trading strategies without secure knowledge of future events. For example, an investor has an initial wealth of 1000 dollars and wishes to convert all dollars to yen over a specified number of days. Each day the investor is offered an exchange rate and must decide whether to accept it or wait for a better one. Facing with the lack of exchange rates, the traditional investor often uses models based on assumptions about the future distribution of exchange rates, and aim for acceptable results on the average (Lippman & McCall, 1981). These models are called Bayesian approach, which assumes probabilistic distribution of future prices based on historical data. However, in some cases, determining the prior distribution in advance may not be possible, because the historical data may be not unavailable. This difficulty is often more extreme in complex dynamical environments such as financial systems.

In order to overcome this difficulty, the competitive analysis, i.e., a relative performance measure is presented originally by El-Yaniv et al. (1998) to study the online one-way trading problem. They viewed the competitive analysis of online one-way trading problem as a two-person game between the online player and an adversary. The investor as an online player chooses an online algorithm ALG and informs the adversary of his choice. The adversary then chooses an input sequence *I* and design optimal offline algorithm *OPT*. A comparison of the performance of ALG to that of *OPT* for *I* is called competitive ratio. They presented a simple threat-based strategy to be optimal and proved its competitive ratio. Competitive analysis appears to be particularly attractive with regard to financial transactions, since there is no need to rely on statistical modeling of input sequences. Essentially, this is major advantage what competitive analysis offers. From then on, there are many extended research about online one-way trading problem (Chen et al. ,2001; Fujiwara et al.,2011; Chin et al.,2014). Nevertheless, the threat-based strategy of El-Yaniv et al. (1998) has been criticized as being too inflexible, which seeks to minimize risk instead of managing it. The reason is that the risk aversion property of the competitive ratio, which leads to overly defensive algorithms. In reality, there are great need of financial players to manage the risk inherent in their activities, requiring balancing the risk and reward associated with decision making under uncertainty. In some cases, whenever investors do have some side information or partial (statistical) knowledge on the evolution of exchange rate sequences, it would be a terrible waste to ignore it. A risk-reward framework was proposed by Al-Binali (1999), combining competitive analysis with two ingredients, i.e., forecast and reward. Namely, the investors prefer to the risk, specified as the ratio of the competitive ratio of the risk algorithm to the optimal competitive ratio of threat-based strategy, since they can benefit from a forecast of exchange rates. When the forecast comes true, the investors boost performance significantly (Ding et al;2010). While it fails, the investors still keep the risk within the desired tolerance for any exchange rate sequences. In short, they extended the pure competitive analysis to allow the online investors to utilize forecasts, while retaining the natural risk aversion of the competitive ratio.

For the risk management in foreign exchange markets, the option is also considered as a better tool to manage the risks associated with the exchange rate volatility (Chalamandaris and Tsekrekos,2011; Lian et al.,2015; Clark,2015). In this paper, we develop a generalized risk- management framework, where an option is a tradable instrument that investors demand in order to manage their risks associated with future financial obligations, not for speculation. Despite the obvious significance and appeal of options in online one-way trading problem, to the best of our knowledge, only few research has been found concerning the competitive analysis with options (see the conclusions of El-Yaniv et al.,2001). Xu et al (2011) introduced the option against the threat of dropping to the minimum price into traditional competitive analysis. Our main contributions are as follows. First, a risk management framework is offered for online one-way trading problem by introducing American put option with the first price as the strike price. This framework extends pure competitive analysis and allows investors to benefit from options. Second, since the option can help the investors to hedge risk, we extend analysis of Al-Binali (1999) to allow the investors to make twice forecasts. For example, the investor makes a forecast first and then, if it is successful, he makes a second forecast to aim at an even better competitive ratio. Even if the forecasts fail, the option tool could help the investors control the risk of performing too poorly with respect to the optimal offline algorithm.

This paper is organized as follows. In Section 2, we provide a literature review on topics related to the on-line unidirectional currency conversion problem. In Section 3 the one-way trading strategy with an American option based on single forecast is presented. In Section 4, we find the optimal online trading algorithm for option protected one-way trading problem based on two-period forecast. Section 5 provides numerical examples about performance comparison between different trading strategies. Finally, conclusions and future research are proposed.

2 Related literatures

This paper focuses on the research closely related to the online one-way trading problem, while skimming the rich literature on Bayesian approaches, dependent on a prior distribution of prices. The readers who are interested in this topic are referred to Lippman and McCall (1981).

The one-way trading problem, which was introduced by El-Yaniv et al. (1998, 2001), involves selling a fixed amount of a product to a sequence of buyers, with the objective of maximizing the seller’s revenue. Chen et al. (2001) extend the model by examining the one-way trading problem with time-varying price bounds and obtain the unique optimal static online algorithm for the problem. Chin et al. (2014) impose no artificial constraints on both upper and lower price bounds and present near optimal algorithms whose performance depends directly on the price sequence. More recently, although competitive analysis frequently leads to the development of strategies, it is often criticized for being too conservative. Several researches try to solve this problem by adopting alternative performance measures. Fujiwara et al. (2011) provide average-case competitive analysis and derive optimal online algorithms for the one-way trading problem under the assumption that the distribution of upper bound of prices is known. Mohr and Schmidt (2013) compare the performance of online algorithms with different types of future information and point out that more information does not necessarily lead to better performance. Wang et al. (2016) approach the one-way trading problem based on competitive difference analysis and show that their policy, keeping a lower standard deviation, performs better than other policies. On the other hand, some researchers notice that investors are interested in managing the risk. They may ask for more reward for their additional risk. Al-Binali (1997) generalizes risk-reward framework of the competitive analysis, in which investors can develop trading strategies according to their risk tolerance and forecast. Iwama and Yonezawa (2001) extend risk-reward framework by defining investors' forecasts in two respects: increasing to some level or never increasing to some level. They provide the optimal algorithms while allowing make twice rounds of forecasts. Xu et al. (2011) apply option tool to improve the bound estimation in El-Yanivs' strategy by only estimation the upper bound.

The online one-way trading problem can be generalized as the time series search problem in financial markets, i.e., the search for best prices in order to buy and/or sell assets (Mohr et al.,2014). El-Yaniv et al. (2001) study the search problem with constant upper and lower price bounds and find the optimal reservation price policy. When price bounds vary with time, Damaschke et al. (2009) present the optimal deterministic algorithms for the model with both upper and lower bounds decaying and another model with only the upper bound decreases. By introducing a general profit function which increases in price while decreases in time, Xu et al. (2011) extend the search problem and propose two optimal deterministic algorithms for both cases with the duration is either known or unknown beforehand. Moreover, Lorenz et al. (2009) extend the time series search problem to the k-search problem, searching for the *k* highest or lowest prices in a sequence. They present the optimal deterministic and randomized algorithms for both maximum and minimum objectives. Zhang et al. (2011) investigate the general k-search problem by trading multiple items at each period and presenting deterministic online algorithm for the case where the quantity of the item is smaller than the length of the trading horizon. Zhang et al. (2012) further consider the so-called multiple time series search problem in which the products may be stored for some periods or be sold immediately at market prices, and they present three types of online algorithms.

All those amendments mentioned above adhere to the competitive analysis. The objective of this paper is to develop trading strategy combining the forecast with the option tool. With the American put option which takes the first price as the strike price, the strategy constricts the risk of sudden dropping to the minimum price. At the same time, because investors may have some forecast of the future, the strategy contains the investor's risk reference and leads to less conservative online policies.

3 Threat-Based Strategy

In the one-way trading problem, an online investor converts dollars to yen over some period of time and must convert all the remaining dollars at the end of a trading period. It is prohibited to convert already purchased yen back to dollars, i.e., the trade is unidirectional. Assume a scenario where the investor wants to convert dollars *D* for yen *Y* within a given time *T*. There are n trading periods T_i, where $i = 1, 2, ...n$. In each trading period T_i the investor is offered an exchange rate p_i at which he may exchange s_i dollars for yen. M and m denote the upper and lower bounds, respectively, of the exchange rate. For simplicity, it assumes that $D = 1$. Hence, the optimal off-line algorithm is to convert all the dollars at the maximum exchange rate (minimum price). The competitive ratio denoted by r can be defined as

$$r = \sup_{p_i} \frac{\max_{i \le j} p_i}{\sum_{i=1}^{n} s_i p_i} \quad (1)$$

where $\max_{i \le j} p_i$ is the maximum revenue of the offline algorithm and $\sum_{i=1}^{n} s_i p_i$ is the revenue of the online algorithm. The object of online investor is to minimize the competitive ratio to gain more performance of online trading strategy. El-Yaniv et al. (2001) make additional assumptions about the exchange rate sequence, i.e., the online investor knows the bounds *m* and *M*. El-Yaniv et al. give an optimal on-line algorithm, which they call a threat-based strategy. That is, the threat-based strategy converts the minimum number of dollars to yen to achieve the optimal competitive ratio under the threat that the adversary will drop the exchange rate to m and keep it there for the remaining trading periods. In the next section we introduce an optimal option-forecast strategy for it. This strategy is a two-stage threat-based algorithm in an option management framework, the first stage is when the forecast has not come true yet, and the second stage is after the forecast has come true.

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4 Option-forecast strategy in a risk management framework

4.1 The risk management framework with option

We extend the model of Al-Binali (1997) to present a new risk management framework, where an investor is allowed to make a forecast based on option. The American put option with the price of $c>0$ is introduced into the one-way trading problem, where the first trading price is taken as the strike price of option. the strategy constricts the risk of sudden dropping to the minimum price. In Figure 1, if the investor chooses not to forecast, they use the option to keep away the risk of sudden dropping to the minimum price and get optimal competitive ratio of r_1. If the investor wishes to take on a specified amount of forecast, in some situations they will do better than the optimal competitive ratio, and in other situations they will do worse. The key points are (i) that the investor can specify (through a forecast) in which situations they will beat the competitive ratio with option, and (ii) that they can limit how badly they perform when the forecast is not correct. The important point is, however, that the risk can be controlled within the factor of $t \geq 1$, e.g., the risk tolerance factor.

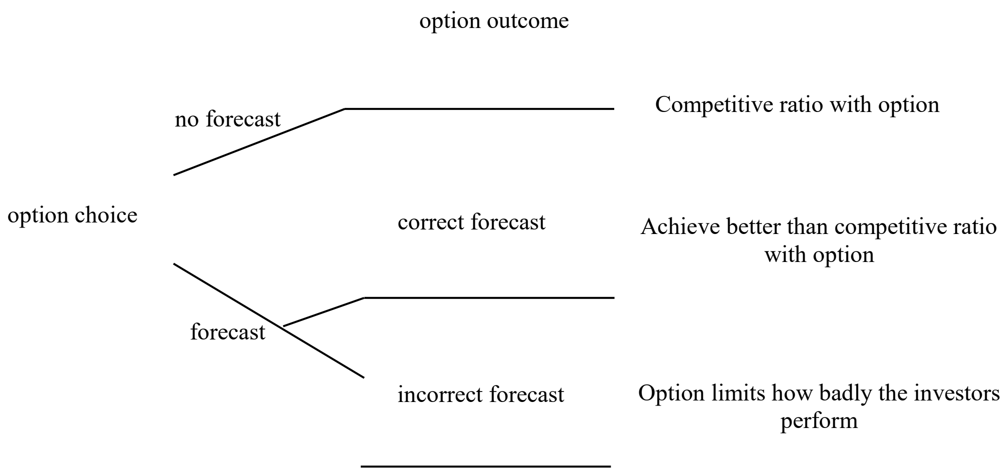

Figure 1 A schematic view of risk management with option

4.2 Option-forecast trading strategy

We now analyze the one-way trading problem in our risk management framework. In the course of trading, forecasts can be made twice or more. By this extension, it is now possible to take several different types of flexible strategies than the threat-based strategy. In some cases, once the exchange rate increases to at least η, then it will remain above η for the rest of trading periods. That is, there exists a price $p_i > \eta$, then the rest of price sequence $p_j > \eta$, $(i < j < n)$. This is the classical momentum effect. In reality of foreign exchange market, when the exchange rate breaks through a certain price, the exchange rate will continue to rise or fall. Hence, when the investor achieves a successful forecast of η_1, he makes a second forecast of η_2 to aim at an even better competitive ratio, i.e., $m < \eta_1 < \eta_2 < M$. That is, if $p_\lambda > \eta_1$, then there may be certain exchange rate, $p_j > \eta_2$, $(\lambda + l \le j < n)$. When $p_i > \eta_1$, the investor will not worry about price dropping to the minimum, and he will not trade, saving dollars for the higher price. Even if the price suddenly drops, the option can decrease the risk of incorrect forecast. Based on this idea, we present the option-forecast trading strategy, denoted by $OFTS$:

The option-forecast trading strategy. Given m, M, c, n, t, a new exchange rate p_i in period T_i and two-stage forecast $m < \eta_1 < \eta_2 < M$, trade according to the following rules:

Rule1. In the first trading period, online investor is offered an exchange rate p_1, he will take $\frac{c}{1+c}$ dollar to buy the amount $\frac{1}{1+c}$ of American put option. Here the striking price of option is p_1 and expiring date will last for the following $n-1$ periods.

Rule 2. From the second period, consider trading dollars to yen only when the current rate is the highest seen so far. In the last trading period, if the exchange rate p_n is larger than p_1, the rest of dollars will be converted into yen, otherwise the option will be exercised at striking price p_1.

Rule 3. While the exchange rate increases to η_1 at the stage of λ, the investor converts enough dollars to ensure a competitive ratio of at most tr_1. He would like to "saving" dollars for the new coming forecast. From stage 2 to stage λ, the amount of trading dollars satisfies $s_i = \frac{1}{tr_1} \frac{p_i - p_{i-1}}{p_i - p_1}$.

Rule 4. When the second forecast η_2 comes true in stage $\lambda + l$, the new game starts and get new optimal competitive ratio r_2. From stage $\lambda + 1$ to stage $\lambda + l - 1$, the investor would like to save dollar and have no trade. When the second forecast come true, the investor only trades enough to guarantee that a competitive ratio of r_2,would be obtained should the exchange rate drop to *m* and remain there for the remainder of the trading periods. I.e., from stage $\lambda + l$ to stage n, the amount of trading dollars satisfies $s_j = \frac{1}{r_2} \frac{p_j - p_{j-1}}{p_j - p_1}$.

Actually, $OFTS$ contains Four stages. In the first stage, the investor purchases the option to decrease the risk of minimum exchange rate. In the secondary stage, the algorithm trades under the threat that the forecast is incorrect, and converts enough dollars to ensure a competitive ratio of tr_1, saving dollars for when the forecast comes true. The third stage begins when the first forecast comes true. The investor will not trade waiting for the second forecast coming true. During the fourth stage, i.e., the second forecast is successful, the algorithm trades to ensure that a competitive ratio of r_2 is achieved. $OFTS$ will be able to make sure that the competitive ratio is no larger than tr_1. When the forecast comes true, more dollars will be saved for higher exchange rates and more yen will be achieved. Hence, $OFTS$ will be more beneficial.

Lemma 1 For $i \geq 1$,

$$s_i = \begin{cases} 0, & \text{when } i = 1 \\ \dfrac{1}{tr_1} \cdot \dfrac{p_2 - \dfrac{tr_1}{1+c} p_1}{p_2 - p_1}, & \text{when } i = 2 \\ \dfrac{1}{tr_1} \cdot \dfrac{p_i - p_{i-1}}{p_i - p_1}, & \text{when } 3 \leq i \leq \lambda - 1 \\ 0, & \text{when } \lambda \leq i \leq \lambda + l - 1 \\ \dfrac{1}{(p_{\lambda+l} - p_1)} \cdot (\dfrac{p_{\lambda+l}}{r_2} - \dfrac{p_{\lambda-1}}{tr_1}), & \text{when } i = \lambda + l \\ \dfrac{1}{r_2} \cdot \dfrac{p_i - p_{i-1}}{p_i - p_1}, & \text{when } \lambda + l + 1 \leq i \leq n \end{cases}$$

Proof. We discuss the trading amount of $OFTS$ from the following four cases.

Case 1: $i = 1$.

The investor has to make choice whether to choose American put option, where the price of the option is c, the striking price is p_1 and expiring date last for $n-1$ periods. According to Rule 1, the investor will take $\frac{c}{1+c}$ dollar to purchase the amount $\frac{1}{1+c}$ of American put option. Hence, the investor can keep the rest amount $\frac{1}{1+c}$ of dollars to exchange dollar into yen during remaining periods. Hence, at this period, the investor will not trade, i.e., $s_1 = 0$.

Case 2: $2 \leq i < \lambda$.

In the stage of $i = 2$, if $p_2 > p_1$, the investor converts just enough dollars to ensure a competitive ratio of at most tr_1 based on Rule 2 and 3. This implies that we have

$$tr_1 = \frac{p_2}{s_2 p_2 + (\dfrac{1}{1+c} - s_2) p_1} \tag{2}$$

From (2), the amount of s_2 is obtained as follows.

$$s_2 = \frac{1}{tr_1} \cdot \frac{p_2 - \frac{tr_1}{1+c} p_1}{p_2 - p_1} \tag{3}$$

During the period of $3 \le i \le l-1$, the first forecast is not realized. Hence, the investor will convert just enough dollars to ensure a competitive ratio of tr_1 under the threat that the price will drop to p_1. We have

$$tr_1 = \frac{p_i}{s_2 p_2 + s_3 p_3 + \ldots + s_i p_i + (\frac{1}{1+c} - s_2 - s_3 - \ldots - s_i) p_1} \tag{4}$$

Hence, we find that

$$s_i = \frac{\frac{1}{tr_1} p_i - s_{i-1}(p_{i-1} - p_1) - \ldots - s_2(p_2 - p_1) - \frac{1}{1+c} p_1}{p_i - p_1} \tag{5}$$

Based on (4), when $i = 3$, the trading amount of s_3 is gained by following equation.

$$s_3 = \frac{\frac{1}{tr_1} p_3 - s_2(p_2 - p_1) - \frac{1}{1+c} p_1}{p_3 - p_1} \tag{6}$$

Combining (2) with (5), we have

$$s_3 = \frac{1}{tr_1} \cdot \frac{p_3 - p_2}{p_3 - p_1} \tag{7}$$

The conductive method is used to analyze the other stages, e.g., s_4, s_5 and so on. Thus, we get following formulas：

$$s_4 = \frac{1}{tr_1} \cdot \frac{p_4 - p_3}{p_4 - p_1}, s_5 = \frac{1}{tr_1} \cdot \frac{p_5 - p_4}{p_5 - p_1}, \ldots, s_{i-1} = \frac{1}{tr_1} \cdot \frac{p_{i-1} - p_{i-2}}{p_{i-1} - p_1} \tag{8}$$

Substitute the s_1, s_2, … , s_{i-1} into (5). That is

$$s_i = \frac{1}{tr_1} \cdot \frac{p_i - p_{i-1}}{p_i - p_1} \tag{9}$$

Case 3: $\lambda \le i \le \lambda + l - 1$.

During the period of $\lambda \le i \le \lambda + l - 1$, the first forecast comes true. It means the exchange price $p_\lambda \ge \eta_1$. According to risk tolerance factor and option factor, the investor would like to expect higher price than η_1, so he will not trade at this period. That is $s_i = 0$.

Case 4: $\lambda + l \le i \le n$.

According to Rule 2, after the second forecast come true, the investor considers trading dollars to yen only when the current rate p_i is the highest seen so far, i.e., from the stage $\lambda + l + 1$ to n. By using the Rule 4, the investor trades so much as to ensure a competitive ratio of r_2 under the threat that the price will drop to p_1. We obtain $Y_i + D_i p_1 = \frac{p_i}{r_2}$ with the following constraints.

$$Y_i = Y_{i-1} + s_i p_i \text{ and } D_i = D_{i-1} - s_i \tag{1}$$

therefore, we get

$$s_i = \frac{1}{r_2} \cdot \frac{p_i - p_{i-1}}{p_i - p_1} (\lambda + 1 \le i \le n) \tag{11}$$

However, in the stage of $i = \lambda + l$, the investor does not believe there are new momentum effect. Hence, he chooses to trade some dollars. Based on equation of (10), we get $Y_{\lambda+l} + p_1 D_{\lambda+l} = Y_{\lambda+l-1} + s_{\lambda+l} p_{\lambda+l} + p_1 D_{\lambda+l-1} - p_1 s_{\lambda+l}$. According to Rule 4, the investor will not trade during period λ and $\lambda + l$, so we gain $Y_{\lambda+l-1} + p_1 D_{\lambda+l-1} = Y_{\lambda-1} + p_1 D_{\lambda-1}$. Combining these two equations, we obtain

$$Y_{\lambda+l} + p_1 D_{\lambda+l} = Y_{\lambda-1} + s_{\lambda+l} p_{\lambda+l} + p_1 D_{\lambda-1} - p_1 s_{\lambda+l} \Rightarrow Y_{\lambda+l} + p_1 D_{\lambda+l} = Y_{\lambda-1} + p_1 D_{\lambda-1} + (p_{\lambda+l} - p_1) s_{\lambda+l} \Rightarrow$$
$$\frac{p_{\lambda+l}}{r_2} = \frac{p_{\lambda-1}}{tr_1} + (p_{\lambda+l} - p_1) s_{\lambda+l} \Rightarrow \quad s_{\lambda+l} = \frac{1}{(p_{\lambda+l} - p_1)} \cdot \left(\frac{p_{\lambda+l}}{r_2} - \frac{p_{\lambda-1}}{tr_1}\right).$$

4.3 The optimal competitive ratio

Lemma 2 For any $k > 2, p_1, p_2$ and p_k

$$\max \sum_{i=3}^{\lambda-1} \frac{p_i - p_{i-1}}{p_i - p_1} = (\lambda - 3)\left(1 - \left(\frac{p_2 - p_1}{p_{l-1} - p_1}\right)^{\frac{1}{\lambda-3}}\right), \ \max \sum_{i=\lambda+2}^{k} \frac{p_i - p_{i-1}}{p_i - p_1} = (k - \lambda - 1)\left(1 - \left(\frac{p_{\lambda+1} - p_1}{p_k - p_1}\right)^{\frac{1}{k-\lambda-1}}\right),$$

and for $3 \le i \le k$,

$$\frac{p_i - p_{i-1}}{p_i - p_1} = 1 - \left(\frac{p_2 - p_1}{p_{\lambda-1} - p_1}\right)^{\frac{1}{\lambda-3}}, \frac{p_i - p_{i-1}}{p_i - p_1} = 1 - \left(\frac{p_{\lambda+1} - p_1}{p_k - p_1}\right)^{\frac{1}{k-\lambda-1}}.$$

Proof. For $3 \le i \le \lambda - 1$, set $x_i = p_i - p_1$. Hence,

$$\sum_{i=3}^{\lambda-1} \frac{p_i - p_{i-1}}{p_i - p_1} = \sum_{i=3}^{\lambda-1} \frac{(p_i - p_1) - (p_{i-1} - p_1)}{p_i - p_1} = (\lambda - 3) - \sum_{i=3}^{\lambda-1} \frac{p_{i-1} - p_1}{p_i - p_1} = (\lambda - 3) - \sum_{i=3}^{\lambda-1} \frac{x_{i-1}}{x_i}$$

Utilizing the geometric-arithmetic mean inequality, we get such characters as follows.

$$\sum_{i=3}^{\lambda-1} \frac{x_{i-1}}{x_i} \ge (\lambda - 3)\left(\frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdot \ldots \cdot \frac{x_{\lambda-2}}{x_{\lambda-1}}\right)^{\frac{1}{\lambda-3}} = (\lambda - 3)\left(\frac{x_2}{x_{\lambda-1}}\right)^{\frac{1}{\lambda-3}}$$

Hence, we have that

$$\max \sum_{i=3}^{\lambda-1} \frac{p_i - p_{i-1}}{p_i - p_1} = (\lambda - 3) - \min \sum_{i=3}^{\lambda-1} \frac{x_{i-1}}{x_i} = (\lambda - 3)\left(1 - \left(\frac{p_2 - p_1}{p_{\lambda-1} - p_1}\right)^{\frac{1}{\lambda-3}}\right) \tag{12}$$

and $\frac{p_i - p_{i-1}}{p_i - p_1} = 1 - \left(\frac{p_2 - p_1}{p_{l-1} - p_1}\right)^{\frac{1}{\lambda-3}}$.

For $\lambda + 2 \le i \le k$, set $y_i = p_i - p_1$, we obtain

$$\sum_{i=\lambda+2}^{k} \frac{p_i - p_{i-1}}{p_i - p_1} = \sum_{i=\lambda+2}^{k} \frac{(p_i - p_1) - (p_{i-1} - p_1)}{p_i - p_1} = (k - \lambda - 1) - \sum_{i=\lambda+2}^{k} \frac{p_{i-1} - p_1}{p_i - p_1} = (k - \lambda - 1) - \sum_{i=\lambda+2}^{k} \frac{y_{i-1}}{y_i}$$

By the geometric-arithmetic mean inequality,

$$\sum_{i=\lambda+2}^{k} \frac{y_{i-1}}{y_i} \ge (k - \lambda - 1)\left(\frac{y_{\lambda+1}}{y_{\lambda+2}} \cdot \ldots \cdot \frac{y_{k-1}}{y_k}\right)^{\frac{1}{k-\lambda-1}} = (k - \lambda - 1)\left(\frac{y_{\lambda+1}}{y_k}\right)^{\frac{1}{k-\lambda-1}}$$

$$\min \sum_{\lambda+2}^{k} \frac{y_{i-1}}{y_i} = (k - \lambda - 1)\left(\frac{y_{\lambda+1}}{y_k}\right)^{\frac{1}{k-\lambda-1}} = (k - \lambda - 1)\left(\frac{p_{\lambda+1} - p_1}{p_k - p_1}\right)^{\frac{1}{k-\lambda-1}}$$

Equality is obtained if and only if all the terms $\frac{y_{i-1}}{y_i}$ in the left-hand side are equal. Hence, we have

$$\max \sum_{i=\lambda+2}^{k} \frac{p_i - p_{i-1}}{p_i - p_1} = (k - \lambda - 1) - \min \sum_{i=\lambda+2}^{k} \frac{y_{i-1}}{y_i} = (k - \lambda - 1)\left(1 - \left(\frac{p_{\lambda+1} - p_1}{p_k - p_1}\right)^{\frac{1}{k-\lambda-1}}\right) \tag{13}$$

and $\frac{p_i - p_{i-1}}{p_i - p_1} = 1 - \left(\frac{p_{\lambda+1} - p_1}{p_k - p_1}\right)^{\frac{1}{k-\lambda-1}}$.

THEOREM 1 For all $1 \le i \le k$, the optimal competitive ratio for $OFTS$ is

$$r_2^{(k)}(p_1) = \frac{(k-\lambda-1)(1-(\frac{\eta_2 - p_1}{p_k - p_1})^{\frac{1}{k-\lambda-1}}) + \frac{\eta_2}{\eta_2 - p_1}}{\frac{tr_1}{(1+c)(tr_1-1)} - \frac{1}{(tr_1-1)} + \frac{1}{\eta_2 - p_1}\frac{p_{\lambda-1}}{tr_1} - \frac{1}{tr_1}(\lambda-3)(1-(\frac{tr_1 p_1 - p_1}{p_{\lambda-1} - p_1})^{\frac{1}{\lambda-3}})}.$$

Proof. For all trading periods of $1 \le i \le k$, the total trading amount is calculated as follows.

$$\sum_{i=1}^{k} s_i = \frac{1}{tr_1}\cdot\frac{p_2 - \frac{tr_1}{1+c}p_1}{p_2 - p_1} + \sum_{i=3}^{\lambda-1}\frac{1}{tr_1}\frac{p_i - p_{i-1}}{p_i - p_1} + \sum_{i=\lambda}^{\lambda+l-1} s_i + \sum_{i=\lambda+l+1}^{k}\frac{1}{r_2}\frac{p_i - p_{i-1}}{p_i - p_1} + \frac{1}{p_{\lambda+l} - p_1}(\frac{p_{\lambda+l}}{r_2} - \frac{p_{\lambda-1}}{tr_1}).$$

Because the investor trades nothing between period λ and $\lambda + l$, we know $\sum_{i=\lambda}^{\lambda+l-1} s_i = 0$.Thus there is only one part associating with l. $\sum_{i=1}^{k} s_i$ increases when l decreases, so when $\sum_{i=1}^{k} s_i$ reaches the maximum, $l = 1$. Hence, we can obtain

$$\sum_{i=1}^{k} s_i = \frac{1}{tr_1}\cdot\frac{p_2 - \frac{tr_1}{1+c}p_1}{p_2 - p_1} + \sum_{i=3}^{\lambda-1}\frac{1}{tr_1}\frac{p_i - p_{i-1}}{p_i - p_1} + \sum_{i=\lambda+2}^{k}\frac{1}{r_2}\frac{p_i - p_{i-1}}{p_i - p_1} + \frac{1}{p_{\lambda+1} - p_1}(\frac{p_{\lambda+1}}{r_2} - \frac{p_{\lambda-1}}{tr_1}).$$

Computing the derivative of $\sum_{i=1}^{k} s_i$ with respect to $p_{\lambda-1}$ or $p_{\lambda+1}$, we obtain

$$\frac{\partial \sum s_i}{\partial p_{\lambda-1}} = \frac{1}{tr_1}[\frac{p_{\lambda-1} - p_1 + p_{\lambda-2} - p_{\lambda-1}}{(p_{\lambda-1} - p_1)^2} - \frac{1}{p_{\lambda+1} - p_1}] = \frac{1}{tr_1}[\frac{(p_{\lambda-2} - p_1)(p_{\lambda+1} - p_1) - (p_{\lambda-1} - p_1)^2}{(p_{\lambda-1} - p_1)^2 (p_{\lambda+1} - p_1)}]$$

Setting $\frac{\partial \sum s_i}{\partial p_{\lambda-1}} = 0$, we have that $(p_{\lambda-2} - p_1)(p_{\lambda+1} - p_1) = (p_{\lambda-1} - p_1)^2$. Then, $\sum_{i=1}^{k} s_i$ has its maximum. Similarly, the derivative of $p_{\lambda-1}$ is calculated by the following equality

$$\begin{aligned}\frac{\partial \sum s_i}{\partial p_{\lambda+1}} &= \frac{1}{r_2}(\frac{1}{p_{\lambda+1} - p_1} - \frac{1}{p_{\lambda+2} - p_1}) - \frac{1}{(p_{\lambda+1} - p_1)^2}(\frac{p_{\lambda+1}}{r_2} - \frac{p_{\lambda-1}}{tr_1}) \\ &\le \frac{1}{r_2}\frac{(p_{\lambda+2} - p_{\lambda+1})(p_{\lambda+1} - p_1) - (p_{\lambda+1} - p_{\lambda-1})(p_{\lambda+2} - p_1)}{(p_{\lambda+1} - p_1)^2 (p_{\lambda+2} - p_1)}\end{aligned}$$

From the stage λ to $\lambda + 1$, the exchange rate will be more than η_2. Because $p_{\lambda+2} - p_1 > p_{\lambda+1} - p_1$, we get $p_{\lambda+1} - p_{\lambda-1} > p_{\lambda+2} - p_{\lambda+1}$, and $\frac{\partial \sum s_i}{\partial p_{\lambda+1}} < 0$. $\sum_{i=1}^{k} s_i$ decreases when $p_{\lambda+1}$ increases. Thus when $p_{\lambda+1}$ is η_2, $\sum_{i=1}^{k} s_i$ has the maximum.

For $\forall k$, the total money for investor is $\sum_{i=1}^{k} s_i \le \frac{1}{1+c}$. If k is known for the investor, then he will spend all the dollars at stage k, and he will get the optimal competitive ratio. Hence, when $\sum_{i=1}^{k} s_i = \frac{1}{1+c}$, we can get

$$\sum_{i=1}^{k} s_i = \frac{1}{1+c} \Rightarrow \frac{p_2}{tr_1(p_2-p_1)} - \frac{p_1}{(1+c)(p_2-p_1)} + \sum_{i=3}^{\lambda-1} \frac{1}{tr_1} \frac{p_i - p_{i-1}}{p_i - p_1} + \sum_{i=\lambda+2}^{k} \frac{1}{r_2} \frac{p_i - p_{i-1}}{p_i - p_1} + \frac{1}{p_{\lambda+1} - p_1}\left(\frac{p_{\lambda+1}}{r_2} - \frac{p_{\lambda-1}}{tr_1}\right) = \frac{1}{1+c}$$

Set $\sum_{i=3}^{\lambda-1} \frac{p_i - p_{i-1}}{p_i - p_1} = x$, $\sum_{i=\lambda+2}^{k} \frac{p_i - p_{i-1}}{p_i - p_1} = y$, we obtain

$$\frac{p_2}{tr_1(p_2-p_1)} + \frac{1}{tr_1}x + \frac{1}{r_2}y + \frac{1}{p_{\lambda+1}-p_1}\left(\frac{p_{\lambda+1}}{r_2} - \frac{p_{\lambda-1}}{tr_1}\right) = \frac{p_2}{(1+c)(p_2-p_1)} \tag{14}$$

Solving (14), we have that

$$r_2 = \frac{y + \frac{p_{\lambda+1}}{p_{\lambda+1}-p_1}}{\frac{p_2}{(1+c)(p_2-p_1)} - \frac{p_2}{tr_1(p_2-p_1)} + \frac{1}{p_{\lambda+1}-p_1}\frac{p_{\lambda-1}}{tr_1} - \frac{1}{tr_1}x} \tag{15}$$

According to Lemma 2 and (15),

$$\max r_2^{(k)}(p_1, p_2, ..., p_k) = r_2^{(k)}(p_1, p_k)$$
$$= \frac{(k-\lambda-1)\left(1-\left(\frac{p_{\lambda+1}-p_1}{p_k-p_1}\right)^{\frac{1}{k-\lambda-1}}\right) + \frac{p_{\lambda+1}}{p_{\lambda+1}-p_1}}{\frac{p_2}{(1+c)(p_2-p_1)} - \frac{p_2}{tr_1(p_2-p_1)} + \frac{1}{p_{\lambda+1}-p_1}\frac{p_{\lambda-1}}{tr_1} - \frac{1}{tr_1}(\lambda-3)\left(1-\left(\frac{p_2-p_1}{p_{\lambda-1}-p_1}\right)^{\frac{1}{\lambda-3}}\right)} \tag{16}$$

Since $p_2 = tr_1 p_1, p_\lambda = \eta_1, p_{\lambda+1} = \eta_2$, we obtain

$$r_2^{(k)}(p_1) = \frac{(k-\lambda-1)\left(1-\left(\frac{\eta_2-p_1}{p_k-p_1}\right)^{\frac{1}{k-\lambda-1}}\right) + \frac{\eta_2}{\eta_2-p_1}}{\frac{tr_1}{(1+c)(tr_1-1)} - \frac{1}{(tr_1-1)} + \frac{1}{\eta_2-p_1}\frac{p_{\lambda-1}}{tr_1} - \frac{1}{tr_1}(\lambda-3)\left(1-\left(\frac{tr_1 p_1 - p_1}{p_{\lambda-1}-p_1}\right)^{\frac{1}{\lambda-3}}\right)} \tag{17}$$

THEOREM 2 For all $1 \le i \le n$, the optimal competitive ratio for $OFTS$ is

$$r_2^{(n)}(p_1) = \frac{(n-\lambda-1)(1-\alpha_1) + \frac{\eta_2}{\eta_2-p_1}}{\frac{tr_1}{(1+c)(tr_1-1)} - \frac{1}{(tr_1-1)} + \frac{1}{\eta_2-p_1}\frac{p_{\lambda-1}}{tr_1} - \frac{1}{tr_1}(\lambda-3)(1-\alpha_2)}$$

where $\alpha_1 = (\frac{\eta_2 - p_1}{M - p_1})^{\frac{1}{n-\lambda-1}}$ and $\alpha_2 = (\frac{tr_1 p_1 - p_1}{p_{\lambda-1} - p_1})^{\frac{1}{\lambda-3}}$.

Proof. Setting $(\frac{\eta_2 - p_1}{p_k - p_1})^{\frac{1}{n-\lambda-1}} = \hat{\alpha}_1$ and $(\frac{tr_1 p_1 - p_1}{p_{\lambda-1} - p_1})^{\frac{1}{\lambda-3}} = \hat{\alpha}_2$, we simplify the equation of (17) and get

$$r_2^{(k)}(p_1) = \frac{(k-\lambda-1)(1-\hat{\alpha}_1) + \frac{\eta_2}{\eta_2 - p_1}}{\frac{tr_1}{(1+c)(tr_1-1)} - \frac{1}{(tr_1-1)} + \frac{1}{\eta_2 - p_1}\frac{p_{\lambda-1}}{tr_1} - \frac{1}{tr_1}(\lambda-3)(1-\hat{\alpha}_2)} \tag{18}$$

Based on (19), $r_2^{(k)}(p_1, p_k)$ decreases when $\hat{\alpha}_1$ increases. $r_2^{(k)}(p_2, p_k)$ is the decreasing function of $\hat{\alpha}_1$, and $\hat{\alpha}_1$. Meanwhile, $r_2^{(k)}(p_2, p_k)$ is the decreasing function of p_k. Thus, r_2 is the increasing function of p_k. When $p_k = M$, $r_2^{(k)}(p_2, p_k)$ has its maximum, and $\hat{\alpha}_1 = \alpha_1 = (\frac{\eta_2 - p_1}{M - p_1})^{\frac{1}{k-\lambda-1}}$. r_2 also have positive relation with k. When $k = n$, we get the maximum of r_2 as follows.

$$r_2^{(n)}(p_1) = \frac{(n-\lambda-1)(1-\alpha_1) + \frac{\eta_2}{\eta_2 - p_1}}{\frac{tr_1}{(1+c)(tr_1-1)} - \frac{1}{(tr_1-1)} + \frac{1}{\eta_2 - p_1}\frac{p_{\lambda-1}}{tr_1} - \frac{1}{tr_1}(\lambda-3)(1-\alpha_2)} \tag{19}$$

Because $p_\lambda = \eta_1$, $p_{\lambda-1}$ will be close to η_1. When $p_{\lambda-1}$ has its maximum $\eta_1 - \xi$ ($\xi \to 0$), we can get the optimal competitive ratio. Setting $p_{\lambda-1} = \eta_1$, we obtain

$$r_2^{(n)}(p_1) = \frac{(n-\lambda-1)(1-\alpha_1) + \frac{\eta_2}{\eta_2 - p_1}}{\frac{tr_1}{(1+c)(tr_1-1)} - \frac{1}{(tr_1-1)} + \frac{1}{\eta_2 - p_1}\frac{\eta_1}{tr_1} - \frac{1}{tr_1}(\lambda-3)(1-\alpha_2)}, \tag{20}$$

where $\alpha_1 = (\frac{\eta_2 - p_1}{M - p_1})^{\frac{1}{n-\lambda-1}}$ and $\alpha_2 = (\frac{tr_1 p_1 - p_1}{p_{\lambda-1} - p_1})^{\frac{1}{\lambda-3}}$.

To get the optimal competitive ratio r_2, we can take the partial derivative of λ, then compute λ when r_2 reaches the maximum, and obtain the optimal competitive ratio.

4 Numerical examples

In this section, we compare the performance of our $OFTS$ to the adaptive threat-based strategy presented by El-Yaniv et al. (2001). Note that if the exchange rate drops all of a sudden, American put option will protect investors, allowing them trade at p_1 instead of the minimum. In general, our strategy performs better during a downtrend. For numerical example, we choose a rising trend of the exchange rate for demonstration. We selected USDJPY data from Nov. 9th, 2016 to Dec. 8th, 2016 in the foreign exchange market. The average exchange rates for 21 days are: 101.9, 105.9, 106.7, 107.5, 107.9, 109, 109.15, 110.6, 110.93, 110.5, 112.65, 113.75, 112.1, 112.05, 112.39, 114.4, 113.73, 113.8, 113.68, 114.25 and 113.4, showed by Figure 1.

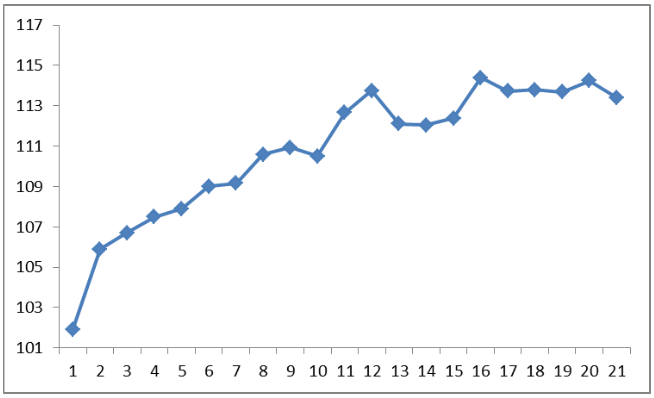

Figure 2 The exchange rate sequence

According to Rule 2, the actual exchange sequence of 13 trading days is: 101.9, 105.9, 106.7, 107.5, 107.9, 109, 109.15, 110.6, 110.93, 112.65, 113.75, 114.4 and 113.4. Assume the investment starts from Nov. 8th, 2016, the investor is planning to exchange our 1 dollar into yens. According to Rule 1, at the first day, the investor would buy an American put option, with a striking price of the first day's exchange rate p_1 = 101.9, to protect against sudden price dropping. The investor would buy option at the first exchange rate S_0 =101.9, the risk-free interest rate is -0.07171% for Japan, 0.28% for the US, the expiration date T = 21/250(assuming 250 trading days per year), volatility σ =0.1 and N =50. Based on CRR tree Model, the value of option is 1.1876 yen, which is 0.011655 dollar at the first day. El-Yaniv's strategy has to make an assumption about the lower and upper bound. Assuming [m, M]=[100,115], the competitive ratio of El-Yaniv's strategy is 1.0502. The competitive ratio of Xu's Strategy is 1.0553. Here we can see that the competitive ratio of Xu's strategy is larger than El-Yaniv's. That is possible when we choose a rising price sequence due to the extra cost of option. The competitive ratio of our strategy is 1.0432 when the risk tolerance $t = 1.01,\ \eta_1 = 110,\ \eta_2 = 112$. For details, we would like to present each period of trading process. Table 1 demonstrates the details for each period of our strategy, compared with El-Yaniv's strategy based on threat.

Table 1 Comparison with El-Yaniv's strategy with each period.

			OFTS ($r_2 = 1.0432$)				El-Yaniv ($r_1 = 1.0502$)			
	Exchange rate		**dollars**		**yen**		**dollars**		**yen**	
2016-MM-DD	Rates Sequence	Rising rates	Dollars traded s_i	Accumulated S	JPY y_i	Accumulated Y	Dollars traded s_i	Accumulated S	JPY y_i	Accumulated Y
11-09	**101.9**	**101.9**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
11-10	105.9	105.9	—	—	—	—	0.1420	0.1420	1.5038	1.5038
11-11	106.7	106.7	—	—	—	—	0.1137	0.2557	12.1318	13.6356
11-14	107.5	107.5	—	—	—	—	0.1016	0.3573	10.9220	24.5576
11-15	107.9	107.9	—	—	—	—	0.0482	0.4055	5.2008	29.7584
11-16	109	109	0.2168	0.2168	23.6312	23.6312	0.1164	0.5219	12.6876	42.4460
11-17	109.15	109.15	0.0194	0.2362	2.1175	25.7487	0.0156	0.5375	1.7027	44.1487
11-18	110.6	110.6	0	—	0	—	0.1303	0.6678	14.4112	58.5599
11-21	110.93	110.93	0	—	0	—	0.0287	0.6965	3.1837	61.7436
11-22	110.5	—	—	—	—	—	—	—	—	—
11-24	112.65	112.65	0.5190	0.7552	58.4654	84.2141	0.1295	0.8260	14.5882	76.3318
11-25	113.75	113.75	0.0890	0.8442	10.1238	94.3379	0.0762	0.9022	8.6678	84.9996
11-28	112.1	—	—	—	—	—	—	—	—	—
11-29	112.05	—	—	—	—	—	—	—	—	—
11-30	112.39	—	—	—	—	—	—	—	—	—
12-01	114.4	114.4	0.0498	0.8940	5.6971	100.0350	0.0430	0.9452	4.9192	89.9188
12-0	113.73	—	—	—	—	—	—	—	—	—

2										
12-05	113.8	—	—	—	—	—	—	—	—	—
12-06	113.68	—	—	—	—	—	—	—	—	—
12-07	114.25	—	—	—	—	—	—	—	—	—
12-08	113.4	113.4	0.0943	0.9883	10.6936	110.7286	0.0548	1	6.2143	96.1331

As we can see from Table 1, our strategy based on two-period forecast and options performs better than El-Yaniv's threat-based strategy. Finally, the investor gets 14.5995 more yen. By our strategy, the investor saves dollars during the first stage of the trading with a risk tolerance of $t = 1.01$. When the first forecast $p_\lambda > \eta_1 = 110$ comes true, the investor is looking for the second price target $\eta_2 = 112$. Once the second forecast becomes true, the investor trades more dollars at a higher exchange rate. Meanwhile, it is obvious that our strategy also outperforms that of El-Yaniv, especially when the exchange rate of the last period is lower than p_1.

5 Conclusions

Online algorithms are designed for the investors who don't have too much information about the price sequences. By designing online strategies, the investors could approach the best offline results even confronted with the worst situation. However, the demand of investors for risk management and preference are always neglected by traditional competitive analysis. We construct the option-forecast trading strategy, which combines both the risk management and risk preference for the investors. The American option with p_1 as the striking price would protect the investors from sudden price dropping and the forecast allows investors to choose risk preference. Compared with former research, the results demonstrate that our strategy performs better, not only constraining the risk of sudden dropping, but also regulating the correspondent online strategy according to the investors' risk preference.

This paper can be extended in several ways. According to behavior finance, the sentiment of the investors will be affected by the market. Hence, it is worthwhile to design the one-way trading strategies with time-varying risk preference of the investors instead of fixed risk tolerance. We also note that considering transaction fees points to another extension of this paper. Furthermore, the two-way trading problem is quite interested, where the investor can buy and sell freely.

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